3.2.1 Exercise: Invertibility and row operations.

Unless otherwise stated, you may use the validity of the result below, labeled ($\star$) (which is yet to be proved, but will be proved very soon):—

($\star$) *Suppose A is a $(p \times p)$-square matrix. Then the statements below are logically equivalent:—*

(a) *A is invertible.*

(b) *A is row-equivalent to I_p.*

(c) *A is a product of $(p \times p)$-row-operation matrices.*

1. For each of the $(p \times p)$-square matrices below, denoted by A here:—
   - obtain, through the application of some appropriate sequence of row operations starting from $\left[\ A \mid I_p\ \right]$, some $(p \times p)$-square matrices $A^\sharp, B^\sharp$ for which $\left[\ A^\sharp \mid B^\sharp\ \right]$ is a row-echelon form row-equivalent to $\left[\ A \mid I_p\ \right]$, and
   - hence decide whether A is invertible.

   Where A is invertible, further obtain the matrix inverse A^{-1} through the application of some appropriate sequence of row operations starting from $\left[\ A^\sharp \mid B^\sharp\ \right]$.

   (a) $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix}$

   (d) $A = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$.

   (f) $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{bmatrix}$.

   (g) $A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$.

   (h) $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & 6 \end{bmatrix}$.

   (i) $A = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 2 & 9 \\ 1 & 1 & -1 & 1 \\ 1 & 2 & 1 & 6 \end{bmatrix}$.

   (j) $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 1 & 2 & 1 & -2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$.

2. Let $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}$ be numbers.

   Let $L(\alpha_{21}, \alpha_{31}, \alpha_{41}, \alpha_{32}, \alpha_{42}, \alpha_{43})$ be the (4×4)-square matrix given by

   $$L(\alpha_{21}, \alpha_{31}, \alpha_{41}, \alpha_{32}, \alpha_{42}, \alpha_{43}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{bmatrix}.$$

   (a) Write down an appropriate sequence of 6 row operations joining an appropriate reduced row-echelon form to $L(\alpha_{21}, \alpha_{31}, \alpha_{41}, \alpha_{32}, \alpha_{42}, \alpha_{43})$

   (b) Hence, or otherwise, show that $L(\alpha_{21}, \alpha_{31}, \alpha_{41}, \alpha_{32}, \alpha_{42}, \alpha_{43})$ is a product of 6 row-operation matrices.

   (c) Is $L(\alpha_{21}, \alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42}, \alpha_{43})$ invertible? Justify your answer, with reference to your answer for the previous part.

3. Let A be an (4×4)-upper triangular matrix, whose (i, j)-th entry is denoted by a_{ij} for each i, j. Suppose all the diagonal entries of A are non-zero.

   (a) Prove that there are some square-matrices B, C such that $A = BC$ and B is a diagonal matrix whose diagonal entries are all non-zero and C is an upper uni-triangular matrix.

   (*Hint.* It may be best to conceive B as a product of some appropriate row-operation matrices.)

   (b) Hence, or otherwise, deduce that A is invertible.

4. Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}$.

   (a) Write down an appropriate sequence of row operations joining A to a reduced row-echelon form which is row-equivalent to A.

   (b) Hence, or otherwise, express A as a product of row-operation matrices.

(c) Is A invertible? Justify your answer, with reference to your answer for the previous part. If A is invertible, also give the matrix inverse of A explicitly.

5. Let A, B, C, D be some (4×4)-square matrices.

Suppose $C = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -6 & 3 & 1 & 0 \\ 24 & -12 & -4 & 1 \end{bmatrix}$, $[\ B \mid C\ ]$ is a reduced row-echelon form, and $[\ A \mid D\ ]$ is row-equivalent to $[\ B \mid C\ ]$, being joint by the sequence of row operations below:

$$[\ A \mid D\ ] \xrightarrow{-2R_1+R_2} \xrightarrow{-1R_2} \xrightarrow{-3R_2+R_3} \xrightarrow{-4R_3+R_4} \xrightarrow{-1R_2+R_1} [\ B \mid C\ ]$$

(a) What is B?

(b) Compute A, D.

(c) Is A invertible? If yes, also give the matrix inverse of A.

6. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$.

(a) Find a reduced row-echelon form which is row-equivalent to $[\ A \mid I_4\ ]$.

(b) Is A invertible? Justify your answer. If A is invertible, also write down the matrix inverse of A.

(c) Let $\mathbf{b}$ be a column vector with 4 entries, whose j-th entry is b_j for each j.

Solve the system $\mathcal{LS}(A,\ \mathbf{b})$, expressing the solutions in terms of b_1, b_2, b_3, b_4.

(d) Let $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Solve the system $\mathcal{LS}\big(A^2,\ \mathbf{e}_2\big)$.

7. Let A be a (4×4)-square matrix. Suppose that there is a sequence of row operations

$$[\ A \mid I_4\ ] \longrightarrow \cdots \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & \beta \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\alpha & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & -\beta & \beta & -\beta & \beta \end{array}\right]$$

in which α, β are some numbers.

(a) Name all possible values of β for which A is invertible.

(b) Suppose A is invertible.

i. Write down the matrix inverse of A. Leave your answer in terms of α, β.

ii. Is it possible for α to be any number?

- If your answer is *yes*, write '*α can be any number*'.
- If your answer is *no*, write down any one number which α cannot equal.

8. Prove the statements below:—

(a) *Let B, C be $(p \times p)$-square matrices. Suppose B, C are invertible. Then B is row-equivalent to C.*

(b) *Let B, C be $(p \times p)$-square matrices. Suppose B is invertible, and B is row-equivalent to C. Then C is invertible.*