3.1.1 Answers to Exercise.

1. (a) $A^2 = \begin{bmatrix} -\beta^2 & \alpha\beta & \beta \\ \alpha\beta & -\alpha^2 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix}$, $A^3 = \mathcal{O}_{3\times 3}$.

   (b) i. ——

   ii. ${C_s}^{-1} = C_{-s}$ for each number s.

2. (a) ——

   (b) *Comment.*

   The argument should be made of two parts. You should argue for both (1) and (2):

   (1) I_n is idempotent and invertible.

   (2) For any $(n \times n)$-square matrix P, if P is idempotent and invertible, then $P = I_n$.

3. (a) $\alpha = 1$, $\beta = -1$.

   (b) *Comment.*

   It will happen that $(A^{-1}B)^2 = I_p$. (So $A^{-1}B$ is involutary.)

4. ——

5. (a) One possible choice is to take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

   (b) One possible choice is to take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

6. ——

7. (a) ——

   (b) An appropriate choice is to take $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

8. ——

9. ——

10. (a) $\alpha = 98$, $\beta = 95$.

    (b) One such choice of a, b, c is given by $a = 1$, $b = -30$, $c = 5$.

    (c) $P^{-1} = -\frac{1}{5}P^2 - \frac{1}{5}P + 6I_3$.

11. (a) $B^2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $B^3 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

    $B^3 - 2B = -I_4$.

    (b) $B^{-1} = -B^2 + 2I_4 = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

    (c) B^t is invertible.

    $(B^t)^{-1} = (B^{-1})^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(d) C is invertible.

$$C^{-1} = B^{-1}(B^t)^{-1} = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 1 & 1 & 0 \\ -2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(e) i. One approrpriate choice for $\widetilde{D}$ is given by $\widetilde{D} = (B + \sqrt{2}I_4)B$.
For such a $\widetilde{D}$, it happens that $D\widetilde{D} = (B - \sqrt{2}I_4)(B + \sqrt{2}I_4)B = B^3 - 2B$.

ii. D is invertible.

$$D^{-1} = -\widetilde{D} = -(B + \sqrt{2}I_4)B = \begin{bmatrix} -1-\sqrt{2} & -1 & -1-\sqrt{2} & 0 \\ 0 & -2+\sqrt{2} & 1-\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} & -1 & 0 \\ 0 & 0 & 0 & -1-\sqrt{2} \end{bmatrix}.$$

12. (a) i. A is invertible, and its matrix inverse is given by $\dfrac{1}{c}A^4 - \dfrac{4}{c}A^2$.

ii. $cI + 4A^3$ invertible, and its matrix inverse is given by $-\dfrac{4}{c^2}A^3 + \dfrac{16}{c^2}A + \dfrac{1}{c}I$.

(b) When $c = 0$, A is not invertible.

13. (a) ——

(b) i. One possible choice of counter-examples is to take $A = B = C = D = I_2$.

ii. One possible choice of counter-examples is to take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

14. ——

15. ——