### 3.1.1 Exercise: Invertible matrices.

In some of the questions below, you will need the respective notions of *idempotency*, *involutoricy*, *polynomial of a square matrix*. Their respective definitions are given below:—

- *Let $C$ be a square matrix.*
  (1) *We say that $C$ is* **idempotent** *if and only if $C^2 = C$.*
  (2) *We say that $C$ is* **involutoric** *if and only if $C^2$ is the identity matrix.*
- *Suppose $B$ is a $(p \times p)$-square matrix, and $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ be numbers.*
  *Then the $(p \times p)$-square matrix given by*
  $$a_0 I_p + a_1 B + a_2 B^2 + \cdots + a_{n-1} B^{n-1} + a_n B^n$$
  *is called the* **polynomial of $B$ with respective coefficients** $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$**.**
  *For convenience of notations, if $f(x)$ is the polynomial with variable $x$ given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$, then we agree to write*
  $$f(B) = a_0 I_p + a_1 B + a_2 B^2 + \cdots + a_{n-1} B^{n-1} + a_n B^n.$$

1. Let $\alpha, \beta$ be numbers. Suppose $\alpha^2 + \beta^2 = 1$.
   Define the $(3 \times 3)$-square matrix $A$ by
   $$A = \begin{bmatrix} 0 & 1 & \alpha \\ -1 & 0 & \beta \\ \alpha & \beta & 0 \end{bmatrix}.$$
   (a) Compute $A^2, A^3$.
   (b) For each number $x$, define the $(3 \times 3)$-square matrix $C_x$ by
   $$C_x = I_3 + xA + \frac{1}{2} x^2 A^2.$$
      i. Show that $C_{s+t} = C_s C_t$ for any numbers $s, t$.
      ii. Hence, or otherwise, show that $C_s$ is invertible for each number $s$.
      What is the matrix inverse of $C_s$?
2. Prove the statements below:—
   (a) *Every involutary matrix is invertible, and its matrix inverse is itself.*
   (b) *$I_n$ is the one and only one idempotent $(n \times n)$-square matrix which is invertible.*
3. Let $A, B$ be $(p \times p)$-square matrices. Suppose $A$ is invertible.
   (a) Verify that $(A+B)A^{-1}(A-B)$, $(A-B)A^{-1}(A+B)$ are equal to each other, and they are equal to $\alpha A + \beta B A^{-1} B$, in which $\alpha, \beta$ are some appropriate integers that you have to name explicitly.
   (b) Suppose $(A+B)A^{-1}(A-B) = \mathcal{O}_{p \times p}$. Show that $B$ is invertible. (*Hint.* First show that $A^{-1}B$ is invertible.)
4. (a) Apply mathematical induction to prove the statement ($\sharp$):—
      ($\sharp$) *Let $A_1, A_2, \cdots, A_n$ be $(p \times p)$-square matrices. Suppose $A_1, A_2, \cdots, A_n$ are invertible.*
      *Then the product $A_1 A_2 \cdots A_n$ is invertible with matrix inverse given by*
      $$(A_1 A_2 \cdots A_n)^{-1} = {A_n}^{-1} \cdots {A_2}^{-1} {A_1}^{-1}.$$
   (b) Hence, or otherwise, prove the statement ($\natural$):—
      ($\natural$) *Let $A$ be a $(p \times p)$-square matrix. Suppose $A$ is invertible.*
      *Then, for each positive integer $n$, the matrix $A^n$ is invertible with matrix inverse given by $(A^n)^{-1} = (A^{-1})^n$.*
5. For each of the statements below, provide a dis-proof against it by giving an appropriate counter-example:—
   (a) *Suppose $A, B$ are invertible $(2 \times 2)$-square matrices. Then $A + B$ is invertible.*
   (b) *Suppose $A, B$ are invertible $(2 \times 2)$-square matrices. Then $A + B$ is not invertible.*

6. Prove the statements below:—

   (a) *Let $A$ be a square matrix. Suppose $A$ is symmetric and $A$ is invertible. Then $A^{-1}$ is symmetric.*

   (b) *Let $A$ be a square matrix. Suppose $A$ is skew-symmetric and $A$ is invertible. Then $A^{-1}$ is skew-symmetric.*

7. (a) Let $A, B$ be $(p \times p)$-square matrices. Suppose $A, B$ are invertible. Further suppose $A, B$ are symmetric, and $A, B$ commute with each other. Prove the statements below:—

      i. *$A^{-1}, B$ commute with each other.*

      ii. *$A, B^{-1}$ commute with each other.*

      iii. *$A^{-1}, B^{-1}$ commute with each other.*

      iv. *$AB$, $A^{-1}B$, $AB^{-1}$, $A^{-}B^{-1}$ are symmetric.*

   (b) Dis-prove the statements below by providing appropriate counter-examples respectively:—

      *Let $A, B$ be $(2 \times 2)$-square matrices. Suppose $A, B$ are invertible. Further suppose $A, B$ are symmetric. Then $A^{-1}B$ is symmetric.*

8. (a) Prove the statement below:—

      i. *Let $A$ be a square matrix. Suppose some entire column of $A$ is a column of zeros. Then $A$ is not invertible.*

      ii. *Let $B$ be a square matrix. Suppose some entire row of $B$ is a row of zeros. Then $B$ is not invertible.*

   (b) Explain why the square matrices below are not invertible.

      i. $\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 3 & 1 & 2 & 0 \\ 3 & 4 & 5 & 0 & 4 & 0 \\ 4 & 5 & 9 & 1 & 8 & 0 \\ 5 & 6 & 7 & 0 & 4 & 0 \\ 6 & 7 & 5 & 1 & 2 & 0 \end{bmatrix}$

      ii. $\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 4 & 2 & 4 & 2 & 4 & 2 \\ 4 & 4 & 4 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 6 & 7 & 3 & 2 & 1 \\ 6 & 7 & 8 & 7 & 8 & 9 \end{bmatrix}$

   (c) Prove the statement ($\sharp$):—

      ($\sharp$) *Let $C, D$ be $(p \times p)$-square matrices. Suppose some column of $CD$ is a column of zeros, or some row of $CD$ is a row of zeros. Then at least one of $C, D$ is not invertible.*

      **Remark.** As a consequence of the statement ($\sharp$), the statement ($\sharp'$) holds:—

      *Let $C, D$ be $(p \times p)$-square matrices. Suppose $CD = \mathcal{O}_{p \times p}$. Then at least one of $C, D$ is not invertible.*

9. Let $A$ be a $(p \times p)$-square matrix, and let $f(x)$ be the polynomial with variable $x$, of degree $n$ greater than 1, given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n$. (So $a_n \neq 0$.)

   Suppose $f(A) = \mathcal{O}_{p \times p}$.

   Suppose $a_0 \neq 0$. Prove that $B$ is invertible. Also write down the matrix inverse of $B$ in terms of the coefficients of the polynomial $f(x)$.

10. Let $P = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & -5 \end{bmatrix}$. Take for granted that $P^2 = \begin{bmatrix} 9 & 19 & 9 \\ 7 & 17 & -8 \\ 0 & -5 & 35 \end{bmatrix}$ and $P^3 = \begin{bmatrix} 46 & 101 & 21 \\ 23 & 38 & \alpha \\ 30 & \beta & -190 \end{bmatrix}$.

    (a) What are the values of $\alpha, \beta$?

    (b) Show that there are some numbers $a, b, c$ such that $P^3 + aP^2 + bP + cI_3 = \mathcal{O}_{3 \times 3}$.

    (c) Show $P$ is invertible, and express the matrix inverse of $P$ in the form $\kappa I_3 + \lambda P + \mu P^2$ for some appropriate numbers $\kappa, \lambda, \mu$.

11. Let $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

    (a) Compute the matrices $B^2$, $B^3$, and $B^3 - 2B$.

    (b) Applying the result of the previous part, or otherwise, show that $B$ is invertible and find its matrix inverse.

    (c) Is $B^t$ invertible? Justify your answer.

       If yes, find its matrix inverse.

    (d) Let $C$ be the square matrix given by $C = B^t B$.

       Is $C$ invertible? Justify your answer.

       If yes, find its matrix inverse.

(e) Let $D$ be the square matrix given by $D = B - \sqrt{2}I_4$.

   i. Write down a square matrix $\widetilde{D}$ in terms of $B$ for which $D\widetilde{D} = B^3 - 2B$. Justify your answer.

   ii. Is $D$ invertible? Justify your answer.
   If yes, find its matrix inverse.

12. Let $A$ be a $(9 \times 9)$-square matrix, and $c$ be a real number.

   Suppose the equality $A^5 - 4A^3 - cI = \mathcal{O}$ holds.

   (Here $I$ stands for the $(9 \times 9)$-identity matrix and $\mathcal{O}$ stands for the $(9 \times 9)$-zero matrix.)

   (a) Suppose $c \neq 0$.

   i. Is $A$ invertible? Justify your answer.

   ii. Is $cI + 4A^3$ invertible? Justify your answer.
   If this matrix is invertible, also express its matrix inverse in the form

$$\alpha I + \beta A + \gamma A^2 + \delta A^3 + \varepsilon A^4$$

   for some appropriate real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$.

   (b) Suppose $c = 0$, and $\frac{1}{2}A$ is not a matrix inverse of itself.
   Is $A$ invertible? Justify your answer.

13. (a) Prove the statements below, with direct reference to the definition of invertibility:—

   i. *Let $A$ be a $(p \times p)$-square matrix, and $D$ be a $(q \times q)$-square matrix. Suppose $A, D$ are invertible. Then the $((p+q) \times (p+q))$-square matrix* $\left[\begin{array}{c|c} A & \mathcal{O}_{p\times q} \\ \hline \mathcal{O}_{q\times p} & D \end{array}\right]$ *is invertible.*

   ii. *Suppose $B$ is a $(p \times q)$-matrix. Then the $((p+q) \times (p+q))$-square matrix* $\left[\begin{array}{c|c} I_p & B \\ \hline \mathcal{O}_{q\times p} & I_q \end{array}\right]$ *is invertible.*

   iii. *Let $A$ be a $(p \times p)$-square matrix, $B$ be a $(p \times q)$-matrix and $D$ be a $(q \times q)$-square matrix. Suppose $A, D$ are invertible. Then the $((p+q) \times (p+q))$-square matrix* $\left[\begin{array}{c|c} A & B \\ \hline \mathcal{O}_{q\times p} & D \end{array}\right]$ *is invertible.*

   **Remark.** For the last statement, one possible approach is to first ask the question what the matrix inverse of $\left[\begin{array}{c|c} A & B \\ \hline \mathcal{O}_{q\times p} & D \end{array}\right]$ will be like if it is invertible. More precisely, what can we say about the $(p \times p)$-square matrix $G$, the $(p \times q)$-matrix $H$, the $(q \times p)$-matrix $K$, and the $(q \times q)$-matrix $L$ if the $((p+q) \times (p+q))$-square matrix $\left[\begin{array}{c|c} G & H \\ \hline K & L \end{array}\right]$ is a matrix inverse of $\left[\begin{array}{c|c} A & B \\ \hline \mathcal{O}_{q\times p} & D \end{array}\right]$?

   (b) Dis-prove the statements below by providing appropriate counter-examples.

   i. *Let $A, B, C, D$ be $(2 \times 2)$-square matrices. Suppose $A, B, C, D$ are invertible. Then* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is invertible.*

   ii. *Let $A, B, C, D$ be $(2 \times 2)$-square matrices. Suppose $A, B, C, D$ are not invertible. Then* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is not invertible.*

14. (a) Prove the statements below, with direct reference to the definition of invertibility:—

   i. *Let $A$ be a $(2 \times 2)$-square matrix whose $(i, j)$-th entry is $a_{ij}$ for each $i, j$. Suppose $A$ is invertible. Then (for each positive integer $n$,)* $\left[\begin{array}{c|c} a_{11}I_n & a_{12}I_n \\ \hline a_{21}I_n & a_{22}I_n \end{array}\right]$ *is invertible.*

   ii. *Let $A$ be a $(2 \times 2)$-square matrix whose $(i, j)$-th entry is $a_{ij}$ for each $i, j$, and $C$ be an $(n \times n)$-square matrix. Suppose $A, C$ are invertible. Then* $\left[\begin{array}{c|c} a_{11}C & a_{12}C \\ \hline a_{21}C & a_{22}C \end{array}\right]$ *is invertible.*

   **Remark.** Note that an assumption that reads '*$A$ is invertible*' allows you to infer that some square matrix $B$ serves as the matrix inverse of $A$. It may be unnecessary to give the entries of $B$ explicitly in terms of the entries of the $A$.

   (b) Prove the statements below, with direct reference to the definition of invertibility:—

   i. *Let $A$ be a $(2 \times 2)$-square matrix whose $(i, j)$-th entry is $a_{ij}$ for each $i, j$. Suppose* $\left[\begin{array}{c|c} a_{11}I_3 & a_{12}I_3 \\ \hline a_{21}I_3 & a_{22}I_3 \end{array}\right]$ *is invertible. Then $A$ is invertible.*

   ii. *Let $A$ be a $(2 \times 2)$-square matrix whose $(i, j)$-th entry is $a_{ij}$ for each $i, j$, and $C$ be $(3 \times 3)$-square matrix. Suppose* $\left[\begin{array}{c|c} a_{11}C & a_{12}C \\ \hline a_{21}C & a_{22}C \end{array}\right]$ *is invertible. Then $A$ is invertible, and $C$ is invertible.*

**Remark.** For the second statement, it may be easier to give a proof-by-contradiction argument. Such an argument should start in this way:—

> *Let $A$ be a $(2 \times 2)$-square matrix whose $(i,j)$-th entry is $a_{ij}$ for each $i, j$, and $C$ be $(3 \times 3)$-square matrix. Suppose $\left[\begin{array}{c|c} a_{11}C & a_{12}C \\ \hline a_{21}C & a_{22}C \end{array}\right]$ is invertible. Further suppose that $A$ was not invertible, or $C$ was not invertible.*

15. Prove the statements (†), (‡) below, with reference to the respective definitions for the notions of left inverse, right inverse:—

    (†) *Let $A$ be a $(p \times q)$-matrix. Suppose $A$ has a left inverse. Then:—*

    (a) *For any column vector $\mathbf{b}$ with $p$ entries, the system $\mathcal{LS}(A, \mathbf{b})$ has at most one solution.*

    (b) *The homogeneous system $\mathcal{LS}(A, \mathbf{0}_p)$ has no non-trivial solution; its only solution is $\mathbf{0}_q$.*

    (‡) *Let $A$ be a $(p \times q)$-matrix. Suppose $A$ has a right inverse. Then, for any column vector $\mathbf{b}$ with $p$ entries, the system $\mathcal{LS}(A, \mathbf{b})$ is consistent.*