2.6.1 Answers to Exercise.

1. (a) Linearly independent.

   (b) Linearly independent.

   (c) Linearly independent.

   (d) Linearly independent.

   (e) Linearly dependent.
   $-3\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}_3$.
   (There are infinitely many non-trivial linear relations.)

   (f) Linearly independent.

   (g) Linearly independent.

   (h) Linearly dependent.
   $2\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}_4{}^t$.
   (There are infinitely many non-trivial linear relations.)

   (i) Linearly independent.

   (j) Linearly independent.

   (k) Linearly independent.

   (l) Linearly dependent.
   $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}_4$.
   (There are infinitely many non-trivial linear relations.)

   (m) Linearly independent.

   (n) Linearly dependent.
   $-2\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}_6$.
   (There are infinitely many non-trivial linear relations.)

2. (a) $\alpha = -6$.
   $-3\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}_5$.

   (b) $\alpha = 2$.
   $\mathbf{u}_1 - 3\mathbf{u}_2 + 2\mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}_4$.

   (c) $\alpha = -2$ and ($\beta = 2$ or $\beta = -2$).
   In each case, $\frac{3}{2}\mathbf{u}_1 - \mathbf{u}_2 - \frac{1}{2}\mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}_5{}^t$.

   (d) $\alpha = -2$ and ($\beta = 2$ or $\beta = -2$).
   In each case, $3\mathbf{u}_1 - 2\mathbf{u}_2 - \mathbf{u}_3 + 2\mathbf{u}_4(\alpha, \beta) = \mathbf{0}_5{}^t$.

3. *Comment.*
   The key is to make use the 'dictionary' between linear combinations and matrix-vector product to translate the assumption

$$\begin{cases} \mathbf{u}_1 = a_{11}\mathbf{t}_1 + a_{21}\mathbf{t}_2 + a_{31}\mathbf{t}_3 + a_{41}\mathbf{t}_4 + a_{51}\mathbf{t}_5 \\ \mathbf{u}_2 = a_{12}\mathbf{t}_1 + a_{22}\mathbf{t}_2 + a_{32}\mathbf{t}_3 + a_{42}\mathbf{t}_4 + a_{52}\mathbf{t}_5 \\ \mathbf{u}_3 = a_{13}\mathbf{t}_1 + a_{23}\mathbf{t}_2 + a_{33}\mathbf{t}_3 + a_{43}\mathbf{t}_4 + a_{53}\mathbf{t}_5 \end{cases}$$

   as

$$\begin{cases} \mathbf{u}_1 = T\mathbf{a}_1, \\ \mathbf{u}_2 = T\mathbf{a}_2, \\ \mathbf{u}_3 = T\mathbf{a}_3 \end{cases}$$

   and further as

$$U = TA,$$

   in which $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$, $T = [\ \mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{t}_3 \mid \mathbf{t}_4 \mid \mathbf{t}_5\ ]$, and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the three columns of A from left to right.