2.6.1 Exercise: Linear dependence and linear independence from the point of view of homogeneous systems of linear equations.

1. For each part below, consider the column/row vectors denoted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$ here.
   - Determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$ are linearly dependent, and
   - when it is, write down a non-trivial linear relation relating $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$.

   (a) $\mathbf{u}_1 = \begin{bmatrix}3\\4\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\-3\end{bmatrix}$.

   (b) $\mathbf{u}_1 = \begin{bmatrix}4\\3\\-2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\-6\\7\end{bmatrix}$.

   (c) $\mathbf{u}_1 = \begin{bmatrix}1\\-2\\4\\3\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}6\\0\\-2\\2\end{bmatrix}$.

   (d) $\mathbf{u}_1 = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$.

   (e) $\mathbf{u}_1 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\1\\-1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}7\\-4\\1\end{bmatrix}$.

   (f) $\mathbf{u}_1 = \begin{bmatrix}1 & 2 & -3\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1 & -3 & 2\end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix}2 & -1 & 5\end{bmatrix}$.

   (g) $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$.

   (h) $\mathbf{u}_1 = \begin{bmatrix}1 & 2 & 3 & 1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}3 & -1 & 2 & 2\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1 & -5 & -4 & 0\end{bmatrix}$.

   (i) $\mathbf{u}_1 = \begin{bmatrix}1 & 4 & 2 & 2 & 6\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1 & 5 & 3 & 3 & 8\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}2 & 6 & 6 & 5 & 11\end{bmatrix}$.

   (j) $\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

   (k) $\mathbf{u}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$.

   (l) $\mathbf{u}_1 = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$.

   (m) $\mathbf{u}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$.

   (n) $\mathbf{u}_1 = \begin{bmatrix}1\\1\\0\\6\\4\\8\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\1\\1\\2\\-2\\5\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}0\\1\\-1\\10\\10\\11\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\-1\\2\\-14\\-16\\-14\end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix}7\\3\\4\\2\\-12\\12\end{bmatrix}$.

2. (a) Let α be a number, and $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$, $\mathbf{u}_3(\alpha) = \begin{bmatrix}1\\-1\\-3\\\alpha+1\\\alpha-1\end{bmatrix}$.

   For which value(s) of α are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, (\alpha)$ are linearly dependent? Justify your answer.
   For each such value of α, write down a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3(\alpha)$.

   (b) Let α be a number, and $\mathbf{u}_1 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$, $\mathbf{u}_2(\alpha) = \begin{bmatrix}1\\1\\\alpha\\3\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}$, $\mathbf{u}_4(\alpha) = \begin{bmatrix}0\\0\\3\\2\alpha^2\end{bmatrix}$.

   For which value(s) of α are $\mathbf{u}_1, \mathbf{u}_2(\alpha), \mathbf{u}_3, \mathbf{u}_4(\alpha)$ are linearly dependent? Justify your answer.
   For each such value of α, write down a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2(\alpha), \mathbf{u}_3, \mathbf{u}_4(\alpha)$.

(c) Let α, β be numbers, and $\mathbf{u}_1 = [\ 1\quad 0\quad -2\quad 1\quad -1\ ]$, $\mathbf{u}_2 = [\ 2\quad 1\quad -2\quad -9\quad 0\ ]$, $\mathbf{u}_3 = [\ 5\quad 8\quad -6\quad 5\quad -1\ ]$, $\mathbf{u}_4(\alpha, \beta) = [\ 3\quad 5\quad \alpha\quad -2\beta^2\quad 1\ ]$.

For which value(s) of α, β are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4(\alpha, \beta)$ are linearly dependent? Justify your answer.

For each such pair of values of α, β, write down a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4(\alpha, \beta)$.

(d) Let α, β be numbers, and $\mathbf{u}_1(\alpha) = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -\alpha \\ 1 \end{bmatrix}$, $\mathbf{u}_2(\beta) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2+\beta \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{u}_3(\beta) = \begin{bmatrix} 0 \\ 3 \\ 9 \\ -3 \\ -9 \\ \beta \end{bmatrix}$, $\mathbf{u}_4(\alpha) = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 8 \\ 1 \\ \alpha^2 \end{bmatrix}$.

For which value(s) of α are $\mathbf{u}_1(\alpha), \mathbf{u}_2(\beta), \mathbf{u}_3(\beta), \mathbf{u}_4(\alpha)$ are linearly dependent? Justify your answer.

For each such pair of values of α, β, write down a non-trivial linear relation amongst $\mathbf{u}_1(\alpha), \mathbf{u}_2(\beta), \mathbf{u}_3(\beta), \mathbf{u}_4(\alpha)$.

3. Let $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$ be column vectors with 9 entries.

Let A be a (5×3)-square matrix, whose (i, j)-th entry is a_{ij}.

For each $j = 1, 2, 3$, define $\mathbf{u}_j = a_{1j}\mathbf{t}_1 + a_{2j}\mathbf{t}_2 + a_{3j}\mathbf{t}_3 + a_{4j}\mathbf{t}_4 + a_{5j}\mathbf{t}_5$.

Suppose $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$ are linearly independent.

Prove that the statements below are logically equivalent:

(1) *The column vectors* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly independent.*

(2) $\mathcal{LS}(A, \mathbf{0}_3)$ *has no non-trivial solution.*