2.6 Linear dependence and linear independence from the point of view of homogeneous systems of linear equations.

0. *Assumed background.*

   - 1.6 *Linear dependence and linear independence.*
   - 2.4 *Solving systems of linear equations.*

   Abstract. We introduce:—

   - a method for systematically determining whether several given column/row vectors are linearly dependent or linearly independent, and finding a non-trivial linear relation when they are linearly dependent.

1. Recall the definition for the notions of *linear dependence* and *linear independence* for column vectors:

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column vectors with p real (or complex) entries.

   (a) *We say that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are* **linearly dependent over the real (or complex) numbers** *if and only if the statement (LD) holds:*

   (LD) *There exist some real (or complex) numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ and $\alpha_1, \alpha_2, \cdots, \alpha_q$ are not all zero.*

   The equality $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ in which $\alpha_1, \alpha_2, \cdots, \alpha_q$ are not all zero is called a **non-trivial linear relation** *of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.*

   (b) *We say that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are* **linearly independent over the real (or complex) numbers** *if and only if the statement (LI) holds:*

   (LI) *For any real (or complex) numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$, if $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_q = 0$.*

   Remark. For simplicity of presentation, we will omit the explicit reference to 'real numbers' or 'complex numbers'. As long as we are consistently thinking in terms of either types of numbers throughout, everything result and argument will work out fine.

2. Recall the result below, labelled Lemma ($\sharp$), which provide a re-formulation for the respective notions of linear dependence and linear independence in terms of homogeneous system of linear equations.

   Lemma ($\sharp$).

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are column vectors with p entries, and U is the $(p\times q)$-matrix given by $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

   Then:—

   (a) *The statements (LD), (LD_0) are logically equivalent:—*

   (LD) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent.*

   (LD_0) *The homogeneous system $\mathcal{LS}(U, \mathbf{0}_p)$ has some non-trivial solution.*

   (b) *The statements (LI), (LI_0) are logically equivalent:—*

   (LI) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly independent.*

   (LI_0) *The homogeneous system $\mathcal{LS}(U, \mathbf{0}_p)$ has no non-trivial solution.*

3. Combined with what we have learnt about (reduced) row-echelon forms and homogeneous systems of linear equations, Lemma ($\sharp$) yields the result below.

   Theorem (1).

   Suppose $p < q$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are column vectors with p entries.

   Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent.

   Remark. In plain words, Theorem (1) says that:—

   any $p+1$ or more column vectors with p entries are definitely linearly dependent.

4. We very often like to re-formulate Theorem (1) in the form of Theorem (2), with a purely logical consideration.

   Theorem (2). (Corollary to Theorem (1).)

   The statements below hold:

   (a) *For each positive integer k, any $p+k$ column vectors with p entries are linearly dependent.*

(b) *Let* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ *be column vectors with* p *entries. Suppose* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ *are linearly independent. Then* $\ell \leq p$.

Remark. After introducing the notions of *basis* and *dimension*, we can generalize Theorem (1) and Theorem (2).

5. **Proof of Theorem (1).**

Suppose $p < q$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are column vectors with p entries.

Define the $(p \times q)$-matrix U by $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

There is some reduced row-echelon form U' so that U' is row-equivalent to U.

U' is an $(p \times q)$-matrix. Then there are at most p non-zero rows in U', and also at most p pivot columns in U'.

Therefore, since $p < q$, there is at least one free column in U'.

Hence $\mathcal{LS}(U', \mathbf{0}_p)$ has a non-trivial solution. And so does $\mathcal{LS}(U, \mathbf{0}_p)$.

Therefore, by Lemma ($\sharp$), $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent.

6. **Question.**

Suppose some column vectors are given to us in 'concrete' terms.

How to determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly dependent or linear independent?*

Answer to the question.

Combined with what we have learnt about solving systems of linear equations, Lemma ($\sharp$) and Theorem (1) suggest the algorithm described below for determining whether they are linearly dependent or linearly independent.

'Algorithm' associated to Lemma ($\sharp$) and Theorem (1).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column vectors with p entries.

We are going to determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent or linearly independent, and to obtain a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ when they are linearly dependent:

Step (0). Ask:—

Is it true that $p < q$?

- If *yes*, conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent. To obtain a non-trivial relation amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ go directly to Step (4).
- If *no*, proceed to Step (1).

Step (1). Form the matrix $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$. Go to Step (2).

Step (2). Obtain some row-echelon form $U^\sharp$ which is row-equivalent to U. Go to Step (3).

Step (3). Inspect $U^\sharp$, and ask:—

Is there some free column in $U^\sharp$?

- If *no*, conclude that the homogeneous system $\mathcal{LS}(U, \mathbf{0})$ has no non-trivial solution.
  Further conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly independent.
- If *yes*, conclude that the homogeneous system $\mathcal{LS}(U, \mathbf{0})$ has some non-trivial solution.
  Further conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent.
  To obtain a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$, go to Step (4).

Step (4). Further obtain from $U^\sharp$, (or directly from U,) a reduced row-echelon form U' which is row-equivalent to U.

Read off from U' a non-trivial solution of $\mathcal{LS}(U, \mathbf{0}_p)$, say, $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$.

Conclude that the equality $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ holds. This equality is a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

7. **Example (1). (Illustrations on the algorithm associated to Lemma ($\sharp$) and Theorem (1).)**

(a) *Let* $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly dependent or linearly independent, and to write down a non-trivial relation amongst* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *when they are linearly dependent.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

We obtain a row-echelon form $U^\sharp$ which is row-equivalent to U:

$$U = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

There is no free column in $U^\sharp$.
Therefore the homogeneous system $\mathcal{LS}(U,\ \mathbf{0}_3)$ has no non-trivial solution.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly dependent or linearly independent, and to write down a non-trivial relation amongst* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *when they are linearly dependent.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

We obtain a row-echelon form $U^\sharp$ which is row-equivalent to U:

$$U = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the 3-rd column of $U^\sharp$ is a free column.
Therefore $\mathcal{LS}(U,\ \mathbf{0}_3)$ has some non-trivial solution.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent.

(What is done next is to obtain a non-trivial linear relation amongst these vectors.)

We further obtain from $U^\sharp$ a reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

U' is the coefficient matrix of the homogeneous system

$$\begin{cases} x_1 \quad\quad + \ x_3 = 0 \\ \quad\quad x_2 - 2x_3 = 0 \\ \quad\quad\quad\quad\quad 0 = 0 \end{cases}$$

Therefore a non-trivial solution of $\mathcal{LS}(U,\ \mathbf{0}_3)$ is given by $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Hence a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is given by $-1 \cdot \mathbf{u}_1 + 2\mathbf{u}_2 + 1 \cdot \mathbf{u}_3 = \mathbf{0}_3$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *are linearly dependent or linearly independent, and to write down a non-trivial relation amongst* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *when they are linearly dependent.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

We obtain a row-echelon form $U^\sharp$ which is row-equivalent to U:

$$U = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is no free column in $U^\sharp$.
Therefore the homogeneous system $\mathcal{LS}(U,\ \mathbf{0}_4)$ has no non-trivial solution.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent.

(d) *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *are linearly dependent or linearly independent, and to write down a non-trivial relation amongst* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *when they are linearly dependent.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

We obtain a row-echelon form $U^\sharp$ which is row-equivalent to U:

$$U = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp = \begin{bmatrix} 1 & 5 & 6 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the 4-th column of $U^\sharp$ is a free column.

Therefore $\mathcal{LS}(U,\ \mathbf{0}_4)$ has some non-trivial solution.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly dependent.

(What is done next is to obtain a non-trivial linear relation amongst these vectors.)

We further obtain from $U^\sharp$ a reduced row-echelon form U' which is row-equivalent to U:

$$U \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

U' is the coefficient matrix of the homogeneous system

$$\left\{ \begin{array}{ccccccc} x_1 & & & - & 2x_4 & = & 0 \\ & x_2 & & + & 4x_4 & = & 0 \\ & & x_3 & - & 3x_4 & = & 0 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \end{array} \right.$$

Therefore a non-trivial solution of $\mathcal{LS}(U,\ \mathbf{0}_4)$ is given by $\begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}$.

Hence a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ is given by $2\mathbf{u}_1 - 4\mathbf{u}_2 + 3\mathbf{u}_3 + 1 \cdot \mathbf{u}_4 = \mathbf{0}_5$.

(e) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 1 \\ 8 \\ 27 \\ 64 \\ 125 \end{bmatrix}$, $\mathbf{u}_6 = \begin{bmatrix} 1 \\ 2 \\ 6 \\ 24 \\ 120 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ *are linearly dependent or linearly independent, and to write down a non-trivial relation amongst* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ *when they are linearly dependent.*

As $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ are six column vectors each with five entries, they are linearly dependent.

(What is done next is to obtain a non-trivial linear relation amongst these vectors.)

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ ]$.

We obtain a reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 4 & 8 & 2 \\ 3 & 5 & 4 & 9 & 27 & 6 \\ 4 & 7 & 8 & 16 & 64 & 24 \\ 5 & 9 & 16 & 25 & 125 & 120 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -70 \\ 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 53 \\ 0 & 0 & 0 & 1 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & -7 \end{bmatrix}.$$

U' is the coefficient matrix of the homogeneous system

$$\left\{ \begin{array}{ccccccccc} x_1 & & & & & - & 70x_6 & = & 0 \\ & x_2 & & + & & + & 8x_6 & = & 0 \\ & & x_3 & & & + & 53x_6 & = & 0 \\ & & & x_4 & & + & 17x_6 & = & 0 \\ & & & & x_5 & - & 7x_6 & = & 0 \end{array} \right.$$

Therefore a non-trivial solution of $\mathcal{LS}(U,\, \mathbf{0}_4)$ is given by $\begin{bmatrix} 70 \\ -8 \\ -53 \\ -17 \\ 7 \\ 1 \end{bmatrix}$.

Hence a non-trivial linear relation amongst $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ is given by $70\mathbf{u}_1 - 8\mathbf{u}_2 + 53\mathbf{u}_3 - 17\mathbf{u}_4 + 7\mathbf{u}_5 + 1 \cdot \mathbf{u}_6 = \mathbf{0}_5$.

8. Lemma ($\sharp$) and Theorem (1) give rise to the results below respectively, which cover the situation for row vectors.

Lemma ($\sharp^*$).

Suppose $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ are row vectors with q entries, and W is the $(p \times q)$-matrix given by $W = \begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \\ \hline \vdots \\ \hline \mathbf{w}_p \end{bmatrix}$.

Then:—

(a) *The statements (LD), (LD^*), (LD_0^*) are logically equivalent:—*

(LD) *The row vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ are linearly dependent.*

(LD^*) *The column vectors $\mathbf{w}_1^t, \mathbf{w}_2^t, \cdots, \mathbf{w}_p^t$ are linearly dependent.*

(LD_0^*) *The homogeneous system $\mathcal{LS}(W^t,\, \mathbf{0}_q)$ has some non-trivial solution.*

(b) *The statements (LI), (LI^*), (LI_0^*) are logically equivalent:—*

(LI) *The row vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ are linearly independent.*

(LI^*) *The column vectors $\mathbf{w}_1^t, \mathbf{w}_2^t, \cdots, \mathbf{w}_p^t$ are linearly independent.*

(LI_0^*) *The homogeneous system $\mathcal{LS}(W^t,\, \mathbf{0}_q)$ has no non-trivial solution.*

Theorem (1^*).

Suppose $p > q$, and $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ are row vectors with q entries.

Then $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ are linearly dependent.

9. Lemma ($\sharp^*$) and Theorem (1^*) inform us how we can adapt the algorithm for determining whether some given column vectors are linearly dependent or linearly independent et cetera to the situation of row vectors.

Example (2). (Illustrations on the application of Lemma ($\sharp^*$) and Theorem (1^*).

(a) *Let* $\mathbf{w}_1 = [\ 1\ \ 1\ \ 2\ \ 1\ \ 3\ ]$, $\mathbf{w}_2 = [\ 2\ \ 3\ \ 6\ \ 1\ \ 2\ ]$, $\mathbf{w}_3 = [\ 2\ \ 3\ \ 5\ \ 1\ \ 1\ ]$, $\mathbf{w}_4 = [\ 4\ \ 5\ \ 6\ \ 3\ \ 4\ ]$.
We want to determine whether $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ are linearly dependent or linearly independent, and to write down a non-trivial relation amongst $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ when they are linearly dependent.

Define $W = \begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \\ \hline \mathbf{w}_3 \\ \hline \mathbf{w}_4 \end{bmatrix}$, and write $U = W^t$.

We obtain a row-echelon form $U^\sharp$ which is row-equivalent to U:

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \\ 1 & 1 & 1 & 3 \\ 3 & 2 & 1 & 4 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the 4-th column of $U^\sharp$ is a free column.
Therefore $\mathcal{LS}(W^t,\, \mathbf{0}_5)$ has some non-trivial solution.
Hence $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ are linearly dependent.
(What is done next is to obtain a non-trivial linear relation amongst $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$.)
We obtain a reduced row-echelon form U' which is row-equivalent to U:

$$U \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

U' is the coefficient matrix of the homogeneous system

$$\begin{cases} x_1 & & & + & 2x_4 & = & 0 \\ & x_2 & & - & 3x_4 & = & 0 \\ & & x_3 & + & 4x_4 & = & 0 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \end{cases}$$

A non-trivial solution of $\mathcal{LS}(W^t,\, \mathbf{0}_5)$ is given by $\begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$.

Hence a non-trivial linear relation amongst $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ is given by $-2\mathbf{w}_1 + 3\mathbf{w}_2 - 4\mathbf{w}_3 + 1 \cdot \mathbf{w}_4 = \mathbf{0}_5^t$.
(Here $\mathbf{0}_5^t$ stands for the zero row vector with five entries.)

(b) *Let* $\mathbf{w}_1 = [\ 1 \quad -1 \quad 3 \quad 2 \quad 1\ ]$, $\mathbf{w}_2 = [\ 2 \quad -1 \quad 4 \quad 3 \quad 3\ ]$, $\mathbf{w}_3 = [\ -5 \quad 3 \quad -10 \quad -8 \quad -4\ ]$, $\mathbf{w}_4 = [\ 15 \quad -9 \quad 31 \quad 25 \quad 13\ ]$.

We want to determine whether $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ *are linearly dependent or linearly independent, and to write down a non-trivial relation amongst* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ *when they are linearly dependent.*

Define $W = \begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \\ \hline \mathbf{w}_3 \\ \hline \mathbf{w}_4 \end{bmatrix}$, and write $U = W^t$.

We obtain a row-echelon form $U^\sharp$ which is row-equivalent to U:

$$U = \begin{bmatrix} 1 & 2 & -5 & 15 \\ -1 & -1 & 3 & -9 \\ 3 & 4 & -10 & 31 \\ 2 & 3 & -8 & 25 \\ 1 & 3 & -4 & 13 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There is no free column in $U^\sharp$.
Therefore the homogeneous system $\mathcal{LS}(W^t,\, \mathbf{0}_5)$ has no non-trivial solution.
Hence $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ are linearly independent.

(c) *Let* $\mathbf{w}_1 = [\ 1 \quad 0 \quad 1 \quad 0 \quad 1\ ]$, $\mathbf{w}_2 = [\ 0 \quad 1 \quad 1 \quad 1 \quad 0\ ]$, $\mathbf{w}_3 = [\ 1 \quad 1 \quad 0 \quad 1 \quad 1\ ]$, $\mathbf{w}_4 = [\ 1 \quad 0 \quad -1 \quad 1 \quad -1\ ]$, $\mathbf{w}_5 = [\ 0 \quad 4 \quad 6 \quad 2 \quad 4\ ]$.

We want to determine whether $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5$ *are linearly dependent or linearly independent, and to write down a non-trivial relation amongst* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5$ *when they are linearly dependent.*

Define $W = \begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \\ \hline \mathbf{w}_3 \\ \hline \mathbf{w}_4 \\ \hline \mathbf{w}_5 \end{bmatrix}$, and write $U = W^t$.

We obtain a row-echelon form $U^\sharp$ which is row-equivalent to U:

$$U = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 4 \\ 1 & 1 & 0 & -1 & 6 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & -1 & 4 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the 5-th column of $U^\sharp$ is a free column.
Therefore $\mathcal{LS}(W^t,\, \mathbf{0}_5)$ has some non-trivial solution.
Hence $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5$ are linearly dependent.
(What is done next is to obtain a non-trivial linear relation amongst $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5$.)
We obtain a reduced row-echelon form U' which is row-equivalent to U:

$$U \longrightarrow \cdots\cdots\cdots \longrightarrow U^\sharp \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

U' is the coefficient matrix of the homogeneous system

$$\begin{cases} x_1 & & & & + & x_5 & = & 0 \\ & x_2 & & & + & 3x_5 & = & 0 \\ & & x_3 & & + & x_5 & = & 0 \\ & & & x_4 & - & 2x_5 & = & 0 \\ & & & & & 0 & = & 0 \end{cases}$$

A non-trivial solution of $\mathcal{LS}(W^t,\, \mathbf{0}_5)$ is given by $\begin{bmatrix} -1 \\ -3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$.

Hence a non-trivial linear relation amongst $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5$ is given by $-\mathbf{w}_1 - 3\mathbf{w}_2 - \mathbf{w}_3 + 2\mathbf{w}_4 + 1 \cdot \mathbf{w}_5 = \mathbf{0}_5^t$. (Here ${\mathbf{0}_5}^t$ stands for the zero row vector with five entries.)