2.5.1 Answers to Exercise.

1. (a) Yes.
   $\mathbf{v} = -6\mathbf{u}_1 + 3\mathbf{u}_2 + 2\mathbf{u}_3$.
   (This is the only possible way to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   (b) No.

   (c) Yes.
   $\mathbf{v} = \mathbf{u}_1 + 3\mathbf{u}_2 - 2\mathbf{u}_3$.
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   (d) Yes.
   $\mathbf{v} = \mathbf{u}_1 - 4\mathbf{u}_2 + \mathbf{u}_3$.
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   (e) Yes.
   $\mathbf{v} = \frac{3}{2}\mathbf{u}_1 + 2\mathbf{u}_2 + \frac{1}{2}\mathbf{u}_3$.
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   (f) Yes.
   $\mathbf{v} = \frac{17}{6}\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 + 0 \cdot \mathbf{u}_3 + \frac{1}{2}\mathbf{u}_4$.
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.)

   (g) Yes. $2\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3 = \mathbf{v}$
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   (h) Yes. $-\frac{9}{4}\mathbf{u}_1 - \frac{1}{4}\mathbf{u}_2 + 4\mathbf{u}_3 = \mathbf{v}$
   (There are infinitely many possible ways to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

2. (a) $\alpha = -8$ only.
   $\mathbf{v}(-8) = -\mathbf{u}_1 + 2\mathbf{u}_2$.

   (b) $\alpha = -6$ only.
   $\mathbf{v}(-6) = \frac{11}{5}\mathbf{u}_1 - \frac{3}{5}\mathbf{u}_2$.

   (c) $\alpha = 4$ only.
   $-\mathbf{u}_1 + 2\mathbf{u}_2 = \mathbf{v}(4)$.

   (d) α can be any number not equal to -1.
   When $\alpha \neq -1$, $\mathbf{v} = \frac{1}{1+\alpha}\mathbf{u}_1 + \frac{1}{1+\alpha}\mathbf{u}_2 - \frac{1}{1+\alpha}\mathbf{u}_3(\alpha)$

   (e) $\alpha = 0$ only.
   $\mathbf{v}(0) = 2\mathbf{u}_1 - \mathbf{u}_2$.

   (f) α can be any number not equal to $\frac{1}{2}$.
   When $\alpha \neq \frac{1}{2}$ $\mathbf{v}(\alpha) = \frac{\alpha^2}{2\alpha - 1}\mathbf{u}_1 + \frac{-\alpha^2 + 6\alpha - 3}{2\alpha - 1}\mathbf{u}_2 - \frac{\alpha}{2\alpha - 1}\mathbf{u}_3(\alpha)$.

   (g) For any numbers α, β, $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.
   $\mathbf{v}(\alpha, \beta) = \frac{9\alpha - \beta}{5}\mathbf{u}_1 + \frac{-2\alpha + 3\beta}{5}\mathbf{u}_2 + \frac{-\alpha - \beta}{5}\mathbf{u}_3$.

3. (a) One possible choice of $\mathbf{b}'$ is given by $\mathbf{b}' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

   (b) A choice of $\mathbf{B}$ corresponding to the choice of $\mathbf{b}'$ above is given by $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   (Such a $\mathbf{b}$ is obtained when we apply on $\mathbf{b}'$ the sequence of reverse row operations

   $$-3R_2 + R_4 - 2R_1 + R_3, 1R_1 + R_2, R_1 \leftrightarrow R_2$$

   that 'un-does' what the sequence of row operations joining A to A' will do to arbitrary matrices with 4 rows.)