2.5.1 Exercise: Linear combinations from the point of view of systems of linear equations.

1. For each part below, consider the column/row vectors denoted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$ and $\mathbf{v}$ here.
   - Determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$, and
   - when it is, write down a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$.

   (a) $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}2\\-1\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}1\\-2\\5\end{bmatrix}$.

   (b) $\mathbf{u}_1 = \begin{bmatrix}1\\-3\\2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\-4\\-1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\-5\\7\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}2\\-5\\3\end{bmatrix}$.

   (c) $\mathbf{u}_1 = \begin{bmatrix}1\\2\\0\\3\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}2\\-1\\2\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}3\\9\\-4\\-2\end{bmatrix}$.

   (d) $\mathbf{u}_1 = \begin{bmatrix}1\\3\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\2\\1\\-1\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}7\\5\\3\\-5\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\-1\\1\\2\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}-1\\9\\0\\0\end{bmatrix}$.

   (e) $\mathbf{u}_1 = [\ 2 \quad 4 \quad 6\ ]$, $\mathbf{u}_2 = [\ -1 \quad 1 \quad 0\ ]$, $\mathbf{u}_3 = [\ 6 \quad 10 \quad 20\ ]$, $\mathbf{v} = [\ 4 \quad 13 \quad 19\ ]$.

   (f) $\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-2\\3\\9\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}3\\1\\3\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}1\\-2\\4\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}2\\1\\4\end{bmatrix}$.

   (g) $\mathbf{u}_1 = [\ 1 \quad 2 \quad 2 \quad 1\ ]$, $\mathbf{u}_2 = [\ 3 \quad 5 \quad 4 \quad 5\ ]$, $\mathbf{u}_3 = [\ 2 \quad 3 \quad -1 \quad 7\ ]$, $\mathbf{v} = [\ 3 \quad 5 \quad -2 \quad 11\ ]$.

   (h) $\mathbf{u}_1 = [\ 1 \quad 2 \quad 7 \quad 1 \quad -1\ ]$, $\mathbf{u}_2 = [\ 3 \quad 2 \quad 5 \quad -1 \quad 9\ ]$, $\mathbf{u}_3 = [\ 1 \quad 1 \quad 3 \quad 1 \quad 0\ ]$, $\mathbf{v} = [\ 1 \quad -1 \quad -5 \quad 2 \quad 0\ ]$.

2. (a) Let α be a number, and let $\mathbf{u}_1 = \begin{bmatrix}3\\0\\-2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\-1\\-5\end{bmatrix}$, $\mathbf{v}(\alpha) = \begin{bmatrix}1\\-2\\\alpha\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}(\alpha)$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2$? Justify your answer.

   (b) Let α be a number, and let $\mathbf{u}_1 = \begin{bmatrix}1\\-3\\2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\-1\\1\end{bmatrix}$, $\mathbf{v}(\alpha) = \begin{bmatrix}1\\\alpha\\5\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}(\alpha)$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2$? Justify your answer.

   (c) Let α be a number, and let $\mathbf{u}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{u}_2(\alpha) = \begin{bmatrix}2\\3\\\alpha\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}3\\4\\5\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2(\alpha)$? Justify your answer. For such value(s) of α, also write down a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2(\alpha)$.

   (d) Let α be a number, and let $\mathbf{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{u}_3(\alpha) = \begin{bmatrix}0\\1\\\alpha\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3(\alpha)$? Justify your answer. For such value(s) of α, also write down a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3(\alpha)$.

   (e) Let $\mathbf{u}_1 = [\ 1 \quad 1 \quad 2\ ]$, $\mathbf{u}_2 = [\ 1 \quad 2 \quad 3\ ]$, $\mathbf{u}_3 = [\ 1 \quad 3 \quad 4\ ]$, $\mathbf{v}(\alpha) = [\ 1 \quad \alpha \quad 1\ ]$.

   For which value(s) of α is $\mathbf{v}(\alpha)$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$? Justify your answer. For such value(s) of α, also write down a linear relation relating $\mathbf{v}(\alpha)$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

   (f) Let α be a number, and let $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\\2\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_3(\alpha) = \begin{bmatrix}0\\0\\\alpha\\1\end{bmatrix}$, $\mathbf{v}(\alpha) = \begin{bmatrix}3\\3\\0\\\alpha\end{bmatrix}$.

   For which value(s) of α is $\mathbf{v}(\alpha)$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3(\alpha)$? Justify your answer. For such value(s) of α, also write down a linear relation relating $\mathbf{v}(\alpha)$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3(\alpha)$.

   (g) Let α, β be numbers, and let $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\\3\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\2\\3\\6\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}2\\5\\3\\10\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}0\\2\\-1\\1\end{bmatrix}$, $\mathbf{v}(\alpha, \beta) = \begin{bmatrix}\alpha\\0\\\beta\\\alpha+\beta\end{bmatrix}$.

   For which value(s) of α, β is $\mathbf{v}(\alpha, \beta)$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$? Justify your answer. For such value(s) of α, β, also write down a linear relation relating $\mathbf{v}(\alpha, \beta)$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

3. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ be column vectors with 4 entries, and $A = [\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_5\ ]$.

Suppose the reduced row-echelon form A', given by

$$A' = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

is resultant from the application on A of the sequence of row operations $R_1 \leftrightarrow R_2$, $-1R_1 + R_2$, $2R_1 + R_3$, $3R_2 + R_4$:

$$A \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{-1R_1+R_2} \xrightarrow{2R_1+R_3} \xrightarrow{3R_2+R_4} A'$$

(a) Write down a column vector $\mathbf{b}'$ with 4 entries for which $\mathcal{LS}(A', \mathbf{b}')$ is inconsistent. Justify your answer.

(b) Hence, or otherwise, find a column vector $\mathbf{b}$ with 4 entries which is not a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$. Justify your answer.