2.5 Linear combinations from the point of view of systems of linear equations.

0. *Assumed background.*

   - 1.5 *Linear combinations.*
   - 2.4 *Solving systems of linear equations.*

Abstract. We introduce:—

- a necessary and sufficient condition, in terms of systems of linear equations, for a given column/row vector to be a linear combination of a number of given column/row vectors, and
- a method for systematically determining whether a given column/row vector is a linear combination of a number of given column/row vectors, and finding a corresponding linear relation when it is.

1. Recall the definition for the notion of *linear combination* for column vectors:

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column vectors with p real (or complex) entries.

   Let $\mathbf{v}$ be a column vector with p real (or complex) entries.

   We say $\mathbf{v}$ is a **linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ over the real (or complex) numbers** *if and only if the statement (LC) holds:*

   (LC) *There exist some real (or complex) numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q$.*

   The expression $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q$ on its own is called the **linear combination of the column vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ with respect to the real (or complex) scalars** *$\alpha_1, \alpha_2, \cdots, \alpha_q$.*

   The equality '$\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q$' is called a **linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$**.

   Remark. For simplicity of presentation, we will omit the explicit reference to 'real numbers' or 'complex numbers'. As long as we are consistently thinking in terms of either types of numbers throughout, everything result and argument will work out fine.

2. Also recall the result below, labelled Lemma $(\star)$, which describes a 'dictionary' between linear combinations of column vectors and matrix-vector products:—

   Lemma $(\star)$.

   Let A be an $(p \times q)$-matrix and $\mathbf{t}$ be a column vector with q entries.

   Suppose that for each $j = 1, 2, \cdots, q$, the j-th column of A is $\mathbf{a}_j$ and the j-th entry of $\mathbf{t}$ is t_j.

   (So $A = [\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_q\ ]$ and $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_q \end{bmatrix}$.)

   Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_q\mathbf{a}_q$.

3. **Question.**

   Suppose the column vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ with p entries are given to us in 'concrete' terms.

   How to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ or not?

   How do we approach this question?

   Lemma $(\star)$ will be instrumental in answering this question.

   With the help of Lemma $(\star)$, we are going to translate this question about a collection of column vectors into a question about a system of linear equations determined by these columns vectors.

   We can answer the latter question immediately and completely. Then with the help of Lemma $(\star)$ again, we will translate the answer to the latter question back into a complete answer to the original question.

   Answer to the question.

   This is provided by Theorem (1) and Theorem (2).

4. **Theorem (1). (Necessary and sufficient condition for a given column vector being a linear combination of given column vectors, in terms of systems of linear equations.)**

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ are column vectors with p entries, and U is the $(p\times q)$-matrix given by $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$. Then the statements $(\dagger), (\dagger_0)$ are logically equivalent:—

($\dagger$) $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *with respect to scalars* $\alpha_1, \alpha_2, \cdots, \alpha_q$.

($\dagger_0$) *The system* $\mathcal{LS}(U, \mathbf{v})$ *is consistent, with a solution* $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$.

Remark. When we do not mention the α_j's, the conclusion in Theorem (1) gives:—

The statements ($\ddagger$), ($\ddagger_0$) *are logically equivalent:—*

($\ddagger$) $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

($\ddagger_0$) *The system* $\mathcal{LS}(U, \mathbf{v})$ *is consistent.*

5. With a purely logical consideration, we obtain Theorem (2) immediately from Theorem (1) as its corollary:—

Theorem (2). (Corollary to Theorem (1).)

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ *are column vectors with* p *entries, and* $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$. *Then the statements* ($\sim\ddagger$), ($\sim\ddagger_0$) *are logically equivalent:—*

($\sim\ddagger$) $\mathbf{v}$ *is not a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

($\sim\ddagger_0$) *The system* $\mathcal{LS}(U, \mathbf{v})$ *is inconsistent.*

6. **Proof of Theorem (1).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ be column vectors with p entries, and $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

(a) Suppose the statement ($\dagger$) holds:

$\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ with respect to scalars $\alpha_1, \alpha_2, \cdots, \alpha_q$.

[We want to deduce the statement ($\dagger_0$): '*The system* $\mathcal{LS}(U, \mathbf{v})$ *is consistent, with a solution* $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$.']

By assumption, $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{v}$.

Define $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$.

Then by Lemma ($\star$), $U\mathbf{t} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{v}$.

Therefore $\mathbf{t}$ is a solution of the system $\mathcal{LS}(U, \mathbf{v})$. By definition, $\mathcal{LS}(U, \mathbf{v})$ is consistent.

Hence the statement ($\dagger_0$) holds.

(b) Suppose the statement ($\dagger_0$) holds:

The system $\mathcal{LS}(U, \mathbf{v})$ is consistent, with a solution $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$.

[We want to deduce the statement ($\dagger$): '$\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *with respect to scalars* $\alpha_1, \alpha_2, \cdots, \alpha_q$.']

Define the column vector $\mathbf{t}$ by $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$.

By definition, $\mathbf{v} = U\mathbf{t}$.

Then, by Lemma ($\star$), $\mathbf{v} = U\mathbf{t} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q$.

Therefore, by definition, $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ with respect to scalars $\alpha_1, \alpha_2, \cdots, \alpha_q$.

Hence the statement ($\dagger$) holds.

7. Theorem (1) and Theorem (2), combined with what we know about solving equations, suggest an 'algorithm' for determining whether a 'concretely' given column vector is a linear combination of a 'concretely' given collection of column vectors.

'Algorithm' associated to Theorem (1) and Theorem (2).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ be column vectors with p entries. We are going to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$, and to obtain a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ when it is:

Step (1). Form the matrix $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

Then form the augmented matrix representation $C = [\ U \mid \mathbf{v}\ ]$ for the system $\mathcal{LS}(U, \mathbf{v})$. Go to Step (2).

Step (2). Obtain some row-echelon form $C^\sharp$ which is row-equivalent to C. Go to Step (3).

Step (3). Inspect $C^\sharp$, and ask:—

Is the last column of $C^\sharp$ a free column?

- If *no*, then conclude that $\mathcal{LS}(U, \mathbf{v})$ is inconsistent.
  Further conclude that $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
- If *yes*, then conclude that $\mathcal{LS}(U, \mathbf{v})$ is consistent.
  Further conclude that $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
  To obtain a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$, go to Step (4).

Step (4). Further obtain from $C^\sharp$ a reduced row-echelon form C' which is row-equivalent to C.

Read off from C' a solution of $\mathcal{LS}(U, \mathbf{v})$, say, $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$.

Conclude that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q$.

8. **Example (1). (Illustrations on the algorithm associated to Theorem (1) and Theorem (2).)**

   (a) *Let* $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$.

   We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, *and to write down a linear relation relating* $\mathbf{v}$ *with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *when it is.*

   Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

   The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is

$$C = \left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{array}\right].$$

   We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right].$$

   Note that the last column of $C^\sharp$ is a free column. Then $\mathcal{LS}(U, \mathbf{v})$ is consistent (and $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$).

   (What is done next is to obtain a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

   We further obtain from $C^\sharp$ a reduced row-echelon form C' which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

   C' is the augmented matrix representation of the system

$$\left\{\begin{array}{rcl} x_1 & = & -3 \\ x_2 & = & 5 \\ x_3 & = & 2 \end{array}\right.$$

   A solution of $\mathcal{LS}(U, \mathbf{v})$ is given by $\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$.

   Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, with a linear relation given by $\mathbf{v} = -3\mathbf{u}_1 + 5\mathbf{u}_2 + 2\mathbf{u}_3$.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 6 \\ -7 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -7 \\ -18 \\ 23 \\ 7 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -23 \\ -55 \\ 73 \\ 33 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$*, and to write down a linear relation relating* $\mathbf{v}$ *with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *when it is.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is

$$C = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -7 & -23 \\ 3 & -2 & 6 & -18 & -55 \\ -4 & 3 & -7 & 23 & 73 \\ 1 & 2 & 0 & 7 & 33 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -7 & -23 \\ 0 & 1 & 0 & 3 & 14 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

(What is done next is to obtain a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.)

We further obtain from $C^\sharp$ a reduced row-echelon form C' which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

C' is the augmented matrix representation of the system

$$\begin{cases} x_1 & & & & = & 1 \\ & x_2 & & & = & 2 \\ & & x_3 & & = & 3 \\ & & & x_4 & = & 4 \end{cases}$$

A solution of $\mathcal{LS}(U, \mathbf{v})$ is given by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, with a linear relation given by $\mathbf{v} = 1 \cdot \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3 + 4\mathbf{u}_4$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 6 \\ 3 \\ -6 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -2 \\ -3 \\ -4 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ 5 \\ 6 \\ -6 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 21 \\ 38 \\ 33 \\ -42 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$*, and to write down a linear relation relating* $\mathbf{v}$ *with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ *when it is.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is

$$C = \left[\begin{array}{ccccc|c} 1 & 3 & -2 & 3 & 21 & 0 \\ 2 & 6 & -3 & 5 & 38 & 0 \\ 1 & 3 & -4 & 6 & 33 & 0 \\ -2 & -6 & 3 & -6 & -42 & 1 \end{array}\right]$$

We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{ccccc|c} 1 & 3 & -2 & 3 & 21 & 0 \\ 0 & 0 & 1 & -1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Note that the last column of $C^\sharp$ is not a free column. Then $\mathcal{LS}(U, \mathbf{v})$ is inconsistent.

Therefore $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$.

(d) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, *and to write down a linear relation relating* $\mathbf{v}$ *with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *when it is.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is

$$C = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots C^\sharp = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Note that the last column of $C^\sharp$ is not a free column. Then $\mathcal{LS}(U, \mathbf{v})$ is inconsistent.

Therefore $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

(e) *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 3 \\ 6 \\ -3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -2 \\ -3 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 1 \\ -3 \\ 10 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ -4 \\ -3 \\ -1 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$, *and to write down a linear relation relating* $\mathbf{v}$ *with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ *when it is.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is

$$C = \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -2 & 1 & -3 \\ 1 & 3 & 2 & -3 & -3 & -4 \\ 2 & 6 & 1 & -2 & 10 & -3 \\ -1 & -3 & -3 & 1 & -5 & -1 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -2 & 1 & -3 \\ 0 & 0 & 1 & -1 & -4 & -1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note that the last column of $C^\sharp$ is a free column. Then $\mathcal{LS}(U, \mathbf{v})$ is consistent (and $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$).

(What is done next is to obtain a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$.)

We further obtain from $C^\sharp$ a reduced row-echelon form C' which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

C' is the augmented matrix representation of the system

$$\left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 3x_2 & & & & & + & 9x_5 & = & 0 \\ & & & & x_3 & & & & & = & 1 \\ & & & & & & x_4 & + & 4x_5 & = & 2 \\ & & & & & & & & 0 & = & 0 \end{array}\right. .$$

A solution of $\mathcal{LS}(U, \mathbf{v})$ is given by $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ with a linear relation given by $\mathbf{v} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 1 \cdot \mathbf{u}_3 + 2 \cdot \mathbf{u}_4 + 0 \cdot \mathbf{u}_5$.

(f) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ -6 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ -1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{u}_6 = \begin{bmatrix} -7 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$, *and to write down a linear relation relating* $\mathbf{v}$ *with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$ *when it is.*

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ ]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is

$$C = \left[\begin{array}{cccccc|c} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{cccccc|c} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note that the last column of $C^\sharp$ is a free column. Then $\mathcal{LS}(U, \mathbf{v})$ is consistent (and $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$).

(What is done next is to obtain a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$.)

We further obtain from $C^\sharp$ a reduced row-echelon form C' which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

C' is the augmented matrix representation of the system

$$\left\{\begin{array}{rcrcccccrcrcr} x_1 & - & 2x_2 & & & & & & & + & x_6 & = & 1 \\ & & & & x_3 & & & + & x_5 & - & 2x_6 & = & 3 \\ & & & & & & x_4 & + & x_5 & - & x_6 & = & 2 \\ & & & & & & & & & & 0 & = & 0 \end{array}\right.$$

A solution of $\mathcal{LS}(U, \mathbf{v})$ is given by $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$, with a linear relation given by $\mathbf{v} = 1 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 3\mathbf{u}_3 + 2\mathbf{u}_4 + 0 \cdot \mathbf{u}_5 + 0 \cdot \mathbf{u}_6$.

9. Now also recall the result below, labelled Lemma ($\star\star$), which describes a'dictionary' between linear combinations of column vectors and that of row vectors:—

Lemma ($\star\star$). ('Dictionary' between linear combinations of column vectors and that of row vectors.)

Suppose $\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are column/row vectors with* p *entries, and* $\alpha_1, \alpha_2, \cdots, \alpha_q$ *are scalars.*

Then the statements below are logically equivalent:

(1) *The column/row vector* $\mathbf{v}$ *is a linear combination of the column/row vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *with respect to scalars* $\alpha_1, \alpha_2, \cdots, \alpha_q$.

(2) *The row/column vector* $\mathbf{v}^t$ *is a linear combination of the row/column vectors* $\mathbf{u}_1^t, \mathbf{u}_2^t, \cdots, \mathbf{u}_q^t$ *with respect to scalars* $\alpha_1, \alpha_2, \cdots, \alpha_q$.

10. Combining Theorem (1), Theorem (2), and Lemma ($\star\star$), we obtain the result below:—-

Theorem (3). (Necessary and sufficient condition for a given row vector being a linear combination of given row vectors, in terms of systems of linear equations.)

Suppose $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p, \mathbf{z}$ *are row vectors with* q *entries, and* $W = \begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \\ \hline \vdots \\ \hline \mathbf{w}_p \end{bmatrix}$. *Then the statements below hold:—*

(a) *The statements* $(\dagger^*), (\dagger_0^*)$ *are logically equivalent:—*

$(\dagger^*)$ $\mathbf{z}$ *is a linear combination of* $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ *with respect to scalars* $\beta_1, \beta_2, \cdots, \beta_p$.

$(\dagger_0^*)$ *The system* $\mathcal{LS}(W^t,\ \mathbf{z}^t)$ *is consistent, with a solution* $\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}$.

(b) *The statements* $(\sim\ddagger^*), (\sim\ddagger_0^*)$ *are logically equivalent:—*

$(\sim\ddagger^*)$ $\mathbf{z}$ *is not a linear combination of* $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$.

$(\sim\ddagger_0^*)$ *The system* $\mathcal{LS}(W^t,\ \mathbf{z}^t)$ *is inconsistent.*

11. **Example (2). (Illustrations on the application of Theorem (3).)**

(a) *Let* $\mathbf{w}_1 = [\ -7\quad 5\quad 1\quad 1\ ]$, $\mathbf{w}_2 = [\ -6\quad 5\quad 0\quad 1\ ]$, $\mathbf{w}_3 = [\ -12\quad 7\quad 4\quad 1\ ]$, $\mathbf{z} = [\ -33\quad 24\quad 5\quad 4\ ]$.
We want to determine whether $\mathbf{z}$ *is a linear combination of* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$*, and to write down a linear relation relating* $\mathbf{z}$ *with* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ *when it is.*

Define $W = \begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \\ \hline \mathbf{w}_3 \end{bmatrix}$.

The augmented matrix representation of $\mathcal{LS}(W^t,\ \mathbf{z}^t)$ is

$$C = \left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \\ 1 & 1 & 1 & 4 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Note that the last column of $C^\sharp$ is a free column. Then $\mathcal{LS}\big(W^t,\ \mathbf{z}^t\big)$ is consistent (and $\mathbf{z}$ is a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$).

(What is done next is to obtain a linear relation relating $\mathbf{z}$ with $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.)

We further obtain from $C^\sharp$ a reduced row-echelon form C' which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

C' is the augmented matrix representation of the system

$$\left\{\begin{array}{rcl} x_1 \qquad\qquad\qquad & = & -3 \\ x_2 \qquad\quad & = & 5 \\ x_3 & = & 2 \\ 0 & = & 0 \end{array}\right.$$

A solution of $\mathcal{LS}(W^t,\ \mathbf{z}^t)$ is given by $\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$.

Then $\mathbf{z}$ is a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, with a linear relation given by $\mathbf{z} = -3\mathbf{w}_1 + 5\mathbf{w}_2 + 2\mathbf{w}_3$.

(b) *Let* $\mathbf{w}_1 = [\ 1\quad 1\quad 3\quad 2\quad 0\ ]$, $\mathbf{w}_2 = [\ 1\quad 0\quad 4\quad 2\quad 2\ ]$, $\mathbf{w}_3 = [\ 1\quad -1\quad 4\quad 1\quad 4\ ]$, $\mathbf{w}_4 = [\ 1\quad 0\quad 3\quad 1\quad 2\ ]$, $\mathbf{z} = [\ 1\quad 0\quad 1\quad 0\quad 1\ ]$.
We want to determine whether $\mathbf{z}$ *is a linear combination of* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$*, and to write down a linear relation relating* $\mathbf{z}$ *with* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ *when it is.*

Define $W = \begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \\ \hline \mathbf{w}_3 \\ \hline \mathbf{w}_4 \end{bmatrix}$.

The augmented matrix representation of $\mathcal{LS}(W^t, \mathbf{z}^t)$ is

$$C = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 1 \\ 2 & 2 & 1 & 1 & 0 \\ 0 & 2 & 4 & 2 & 1 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note that the last column of $C^\sharp$ is not a free column.
Then $\mathcal{LS}(W^t, \mathbf{z}^t)$ is inconsistent.
Therefore $\mathbf{z}$ is not a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$.