2.4.1 Answers to Exercise.

1. (a) Coefficient matrix: $\begin{bmatrix} 0 & 1 & 1 & -1 \\ 2 & -1 & -1 & 1 \\ 3 & 0 & -2 & -2 \end{bmatrix}$.

   Vector of constants: $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

   (b) '$x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$' is a solution of the system (S).

   (c) (S) is consistent.

   (d) (S) has two or more solutions.

2. (a) $A = \begin{bmatrix} 0 & 1 & -2 & 1 & 0 \\ 1 & -1 & 2 & -2 & 2 \\ 2 & 1 & -3 & 2 & -2 \\ 1 & 1 & 2 & -4 & 2 \end{bmatrix}$.

   $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$.

   $C = \left[\begin{array}{ccccc|c} 0 & 1 & -2 & 1 & 0 & 2 \\ 1 & -1 & 2 & -2 & 2 & -1 \\ 2 & 1 & -3 & 2 & -2 & 3 \\ 1 & 1 & 2 & -4 & 2 & -1 \end{array}\right]$.

   (b) '$x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 2$, $x_5 = 1$' is a solution of the system (S).

   (c) i. (S) is not homogeneous.

   ii. (S) is consistent.

   iii. (S) has two or more solutions solutions.

3. (a) $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -2 \\ -1 & 1 & 0 & 2 & 4 \\ 3 & 1 & 2 & 2 & 4 \\ 2 & 0 & 2 & -2 & -6 \end{bmatrix}$.

   (b) $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 6 \\ 0 \end{bmatrix}$

   (c) $\mathcal{LS}(A, \mathbf{b})$ is consistent.

   (d) $g_0 = 1$, $g_1 = g_2 = g_3 = g_4 = g_5 = g_6 = 0$.

   (e) $h_1 = 1$, $h_2 = 1$, $h_3 = -2$, $h_4 = 2$.

4. (a) $C_b = \left[\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ -1 & 2 & 3 & 1 & b \\ 2 & -1 & 0 & 4 & 5 \end{array}\right]$.

   (b)

$$\begin{aligned} C_b &= \left[\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ -1 & 2 & 3 & 1 & b \\ 2 & -1 & 0 & 4 & 5 \end{array}\right] \xrightarrow{1R_1+R_3} \xrightarrow{-2R_1+R_4} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 3 & 6 & 1 & 2+b \\ 0 & -3 & -6 & 4 & 1 \end{array}\right] \\ &\xrightarrow{-3R_2+R_3} \xrightarrow{3R_2+R_4} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & -2 & -4+b \\ 0 & 0 & 0 & 7 & 7 \end{array}\right] \xrightarrow{\frac{1}{7}R_4} \xrightarrow{R_3 \leftrightarrow R_4} \xrightarrow{2R_3+R_4} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2+b \end{array}\right] \\ &\xrightarrow{-1R_2+R_1} \xrightarrow{-1R_3+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2+b \end{array}\right] \end{aligned}$$

   (c) Suppose the system (S_b) is consistent.
   Then $b = 2$.
   A full description of all solutions of the system (S_2) is given by:—

- $\mathbf{t}$ is a solution of (S_2) if and only if there is some number u such that $\mathbf{t} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + u \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

5. (a) The augmented matrix representation of $\mathcal{LS}(A, \mathbf{b})$ is given by $C = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{array}\right]$.

A row-echelon form $C^\sharp$ which is row-equivalent to C is given by $C^\sharp = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{array}\right]$.

The last column of $C^\sharp$ is a free column. Therefore $\mathcal{LS}(A, \mathbf{b})$ is consistent.

A reduced row-echelon form C' which is row-equivalent to C is given by $C' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$.

The one and only one solution of $\mathcal{LS}(A, \mathbf{b})$ is given by $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

(b) The augmented matrix representation of $\mathcal{LS}(A, \mathbf{b})$ is given by $C = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{array}\right]$.

A row-echelon form $C^\sharp$ which is row-equivalent to C is given by $C^\sharp = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$.

The last column of $C^\sharp$ is a pivot column. Therefore $\mathcal{LS}(A, \mathbf{b})$ is inconsistent.

(c) The augmented matrix representation of $\mathcal{LS}(A, \mathbf{b})$ is given by $C = \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{array}\right]$.

A row-echelon form $C^\sharp$ which is row-equivalent to C is given by $C^\sharp = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

The last column of $C^\sharp$ is a free column. Therefore $\mathcal{LS}(A, \mathbf{b})$ is consistent.

A reduced row-echelon form C' which is row-equivalent to C is given by $C' = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if there is some number u such that $\mathbf{t} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

(d) The augmented matrix representation of $\mathcal{LS}(A, \mathbf{b})$ is given by $C = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right]$.

A row-echelon form $C^\sharp$ which is row-equivalent to C is given by $C^\sharp = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The last column of $C^\sharp$ is a free column. Therefore $\mathcal{LS}(A, \mathbf{b})$ is consistent.

A reduced row-echelon form C' which is row-equivalent to C is given by $C' = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if there are some numbers u, v such that

$$\mathbf{t} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

(e) The augmented matrix representation of $\mathcal{LS}(A, \mathbf{b})$ is given by $C = \left[\begin{array}{ccccc|c} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{array}\right]$.

A row-echelon form $C^\sharp$ which is row-equivalent to C is given by $C^\sharp = \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$.

The last column of $C^\sharp$ is a free column. Therefore $\mathcal{LS}(A, \mathbf{b})$ is consistent.

A reduced row-echelon form C' which is row-equivalent to C is given by $C' = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{b})$ if and only if there are some numbers u, v such that

$$\mathbf{t} = \begin{bmatrix} 10 \\ -8 \\ 0 \\ 5 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

(f) The augmented matrix representation of $\mathcal{LS}(A,\, \mathbf{b})$ is given by $C = \left[\begin{array}{cccc|c} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{array}\right]$.

A row-echelon form $C^\sharp$ which is row-equivalent to C is given by $C^\sharp = \left[\begin{array}{cccc|c} 1 & 2 & 7 & 1 & -1 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The last column of $C^\sharp$ is a free column. Therefore $\mathcal{LS}(A,\, \mathbf{b})$ is consistent.

A reduced row-echelon form C' which is row-equivalent to C is given by $C' = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{b})$ if and only if there is some number u such that $\mathbf{t} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -2 \end{bmatrix} + u \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix}$.

(g) The augmented matrix representation of $\mathcal{LS}(A,\, \mathbf{b})$ is given by $C = \left[\begin{array}{cccccc|c} 5 & 5 & 4 & 13 & 9 & 2 & 16 \\ 1 & 1 & 0 & 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 6 & 4 & 1 & 7 \\ 2 & 2 & 1 & 4 & 3 & 1 & 6 \end{array}\right]$.

A row-echelon form $C^\sharp$ which is row-equivalent to C is given by $C^\sharp = \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The last column of $C^\sharp$ is a free column. Therefore $\mathcal{LS}(A,\, \mathbf{b})$ is consistent.

A reduced row-echelon form C' which is row-equivalent to C is given by $C' = \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{b})$ if and only if there are some numbers u, v, w such that

$$\mathbf{t} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + u \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

6. (a) A reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

A full description of the solutions of the homogeneous system $\mathcal{LS}(A,\, \mathbf{0}_2)$ is given by:

- $\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{0}_2)$ if and only if there is some number u such that $\mathbf{t} = u \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$.

(b) A reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

A full description of the solutions of the homogeneous system $\mathcal{LS}(A,\, \mathbf{0}_3)$ is given by:

- $\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{0}_3)$ if and only if there is some number u such that $\mathbf{t} = u \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$.

(c) A reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 0 & -5 & 0 & -3 \\ 0 & 1 & 7 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

A full description of the solutions of the homogeneous system $\mathcal{LS}(A,\, \mathbf{0}_3)$ is given by:

- $\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{0}_3)$ if and only if there are some numbers u, v such that $\mathbf{t} = u \begin{bmatrix} 5 \\ -7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

(d) A reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

A full description of the solutions of the homogeneous system $\mathcal{LS}(A, \mathbf{0}_4)$ is given by:

- $\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{0}_4)$ if and only if there are some numbers u, v, w such that

$$\mathbf{t} = u \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ -5 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

7. (a) $C_{\mathbf{b}} = \begin{bmatrix} 1 & 2 & 1 & 1 & b_1 \\ 2 & 1 & -1 & 3 & b_2 \\ 4 & -1 & -5 & 7 & b_3 \end{bmatrix}$.

(b) $C_{\mathbf{b}}$ is row-equivalent to $C'_{\mathbf{b}} = \begin{bmatrix} 1 & 2 & 1 & 1 & b_1 \\ 0 & 1 & 1 & -1/3 & (-2b_1 + b_2)/3 \\ 0 & 0 & 0 & 0 & 2b_1 - 3b_2 + b_3 \end{bmatrix}$.

(c) *Comment.*

$(S_{\mathbf{b}})$ is consistent if and only if the last column of $C'_{\mathbf{b}}$ is not a pivot column.

(d) $\mathbf{t}$ is a solution if there exists some u, v such that $\mathbf{t} = \begin{bmatrix} (-b_1 - 2b_2)/3 \\ (-2b_1 + b_2)/3 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -5/3 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}$.

8. (a) ——

(b) ——

(c) i. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \beta - 1 \end{bmatrix}$

ii. When $\beta \neq 1$, $\mathbf{t}$ is a solution of $A_{2,\beta}$ if and only if there exists some number c such that $\mathbf{t} = c \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

iii. $\mathbf{t}$ is a solution of $A_{2,1}$ if and only if there exist some c_1, c_2 such that $\mathbf{t} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

9. (a) ——-

(b) One possible choice is given by $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{b} = \mathbf{b}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) i. $\mathbf{0}_3$ is a solution of the system whose augmented matrix representation is D.
$\mathbf{0}_3$ is not a solution of the system whose augmented matrix representation is C.

ii. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is a solution of the system whose augmented matrix representation is C.

$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is not a solution of the system whose augmented matrix representation is D.

(There are many valid answers. As long as a concrete vector $\mathbf{t}$ is named for which $\mathbf{t}$ is a solution of C and not a solution of D, or a concrete vector $\mathbf{u}$ is named for which $\mathbf{u}$ is a solution of D and not a solution of C, the answer is valid.)

iii. $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a solution of the system whose augmented matrix representation is D.

But it is not a solution of the system whose augmented matrix representation is C.