2.4.1. Exercise: Solving systems of linear equations.

In some of the questions below, you will need the notion of *equivalent systems of linear equations.* Its definition is given below:—

- *Let A, A' be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{b}'$ be column vectors with m entries.*

  We say that $\mathcal{LS}(A, \mathbf{b})$ is **equivalent (as a system) to** *$\mathcal{LS}(A', \mathbf{b}')$ if and only if the statement (ES) holds:*

(ES) For any column vector $\mathbf{t}$ with n entries, $\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if $\mathbf{t}$ is a solution of $\mathcal{LS}(A', \mathbf{b}')$.

1. Consider the system of linear equations

$$(S): \left\{\begin{array}{rcrcrcrcr} & & x_2 & + & x_3 & - & x_4 & = & 1 \\ 2x_1 & - & x_2 & - & x_3 & + & x_4 & = & 1 \\ 3x_1 & & & - & 2x_3 & - & 2x_4 & = & -1 \end{array}\right.$$

   (a) Write down the coefficient matrix and the vector of constants for the system (S).

   (b) Is '$x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$' a solution of the system (S)?

   (c) Is (S) consistent or inconsistent?

   (d) Does (S) have one solution only, or two or more solutions, or neither?

2. Consider the system of linear equations

$$(S): \left\{\begin{array}{rcrcrcrcrcr} & & x_2 & - & 2x_3 & + & x_4 & & & = & 2 \\ x_1 & - & x_2 & + & 2x_3 & - & 2x_4 & + & 2x_5 & = & -1 \\ 2x_1 & + & x_2 & - & 3x_3 & + & 2x_4 & - & 2x_5 & = & 3 \\ x_1 & + & x_2 & + & 2x_3 & - & 4x_4 & + & 2x_5 & = & -1 \end{array}\right.$$

   (a) Write down the coefficient matrix A, the vector of constants $\mathbf{b}$, and the augmented matrix representation C, for the system (S).

   (b) Is '$x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 2$, $x_5 = 1$' a solution of the system (S)?

   (c) i. Is (S) homogeneous or not?

   ii. Is (S) consistent or inconsistent?

   iii. Does (S) have one solution only, or two or more solution, or neither?

3. Write $C = \left[\begin{array}{rrrrr|r} 1 & 3 & 1 & -2 & -2 & b_1 \\ -1 & 1 & 0 & 2 & 4 & b_2 \\ 3 & 1 & 2 & 2 & 4 & b_3 \\ 2 & 0 & 2 & -2 & -6 & b_4 \end{array}\right]$, $C' = \left[\begin{array}{ccccc|r} g_0 & g_1 & g_2 & g_3 & h_1 & 2 \\ 0 & 1 & g_4 & g_5 & h_2 & 0 \\ 0 & 0 & 1 & g_6 & h_3 & -1 \\ 0 & 0 & 0 & 1 & h_4 & 1 \end{array}\right]$, in which the b_j's, the g_j's and the h_j's are some numbers.

   Suppose that (I), (II), (III) hold:

   (I) C is the augmented matrix representation for a certain system of linear equations $\mathcal{LS}(A, \mathbf{b})$ (whose matrix of coefficients is A and whose vector of constant is $\mathbf{b}$).

   (II) The vector $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$.

   (III) C' is the reduced row-echelon form which is row equivalent to C.

   (a) Write down A.

   (b) Write down $\mathbf{b}$ (giving the explicit values of its entries).

   (c) Is $\mathcal{LS}(A, \mathbf{b})$ consistent or inconsistent?

   (d) What are the values of $g_0, g_1, g_2, g_3, g_4, g_5, g_6$?

   (e) What are the values of h_1, h_2, h_3, h_4?

4. Consider the following system of linear equations

$$(S_b): \left\{\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & 3x_3 & & & = & 2 \\ & & x_2 & + & 2x_3 & + & x_4 & = & 2 \\ -x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & b \\ 2x_1 & - & x_2 & & & + & 4x_4 & = & 5 \end{array}\right.$$

   in which b is some number.

(a) Write down the augmented matrix representation C_b for the system (S_b).

(b) Show that C_b is row-equivalent to the matrix

$$C_b' = \left[\begin{array}{cccc|c} 1 & 0 & \alpha_1 & 0 & \beta_1 \\ 0 & 1 & \alpha_2 & 0 & \beta_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 \\ 0 & 0 & 0 & \alpha_4 & \beta_4 + b \end{array}\right]$$

in which $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ are some real numbers whose values are independent of b, and for which the matrix

$$\begin{bmatrix} 1 & 0 & \alpha_1 & 0 \\ 0 & 1 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix}$$

is a reduced row-echelon form.
(You are required to give the explicit values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$.)

(c) Suppose the system (S_b) is consistent.
What is the value of b?
Give a full description of the solutions of (S_b).

5. In each part below, solve the system of linear equations with coefficient matrix A and vector of constants $\mathbf{b}$, by

- writing down the augmented matrix representation C of $\mathcal{LS}(A, \mathbf{b})$, and finding a row-echelon form $C^\sharp$ which is row-equivalent to C.
- determining whether the system $\mathcal{LS}(A, \mathbf{b})$ is consistent, and
- finding a reduced row-echelon form C' which is row equivalent to C, and giving a full description of all solutions of the system $\mathcal{LS}(A, \mathbf{b})$, when it is consistent.

(a) $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ 11 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ -2 & -1 & -3 & 3 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

(f) $A = \begin{bmatrix} 1 & 2 & 7 & 1 \\ 1 & 1 & 3 & 1 \\ 3 & 2 & 5 & -1 \\ 1 & -1 & -5 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}$.

(g) $A = \begin{bmatrix} 5 & 5 & 4 & 13 & 9 & 2 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 2 & 2 & 2 & 6 & 4 & 1 \\ 2 & 2 & 1 & 4 & 3 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 16 \\ 2 \\ 7 \\ 6 \end{bmatrix}$.

6. In each part below, solve the homogeneous system of linear equations with coefficient matrix A, by

- finding a reduced row-echelon form A' which is row equivalent to A, and giving a full description of all solutions of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

(a) $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 12 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 3 & -3 & 11 \\ -1 & 3 & 6 & -1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 2 & 9 & 5 & 4 \\ 2 & 1 & -3 & 6 & 1 \\ 5 & 4 & 3 & 12 & 1 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 3 \\ 2 & 4 & 1 & 1 & 2 & 1 \\ 4 & 8 & 2 & 2 & 3 & 5 \end{bmatrix}$.

7. For any numbers b_1, b_2, b_3, write $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, and denote by $(S_\mathbf{b})$ the system of linear equations below:—

$$(S_\mathbf{b}): \left\{ \begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & b_1 \\ 2x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & b_2 \\ 4x_1 & - & x_2 & - & 5x_3 & + & 7x_4 & = & b_3 \end{array} \right.$$

(a) Write down the augmented matrix representation $C_\mathbf{b}$ of the system $(S_\mathbf{b})$.

(b) Show that $C_{\mathbf{b}}$ is row-equivalent to some row-echelon form $C_{\mathbf{b}}^{\sharp}$, given by

$$C_{\mathbf{b}}^{\sharp} = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & \gamma \\ 0 & 1 & \alpha & \beta & \delta \\ 0 & 0 & 0 & 0 & \varepsilon \end{array}\right]$$

in which $\alpha, \beta, \gamma, \delta, \varepsilon$ are some appropriate numbers, possibly dependent on b_1, b_2, b_3. You have to give the values of $\alpha, \beta, \gamma, \delta, \varepsilon$ explicitly.

(c) Show that $(S_{\mathbf{b}})$ is consistent if and only if $b_3 = 3b_2 - 2b_1$.

(d) Suppose $b_3 = 3b_2 - 2b_1$. Give a full description of all solutions of $(S_{\mathbf{b}})$.

8. For any real numbers α, β, define $A_{\alpha,\beta} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & \alpha & \beta \end{bmatrix}$.

Consider the homogeneous system $\mathcal{LS}(A_{\alpha,\beta},\, \mathbf{0}_3)$.

(a) Suppose $\mathcal{LS}(A_{\alpha,\beta},\, \mathbf{0}_3)$ has a non-trivial solution, say, $\begin{bmatrix} a \\ b \\ c \\ -1 \end{bmatrix}$.

Show that $(\alpha - 2)c = \beta - 1$.

(b) Suppose $\alpha \neq 2$.

i. Show that a non-trivial solution of $\mathcal{LS}(A_{\alpha,\beta},\, \mathbf{0}_3)$ is given by

$$\mathbf{s}_{\alpha,\beta} = \begin{bmatrix} 1 - \beta \\ \alpha - \beta - 1 \\ \beta - 1 \\ 2 - \alpha \end{bmatrix}$$

ii. By considering an appropriate row-echelon form or reduced row-echelon form which is row-equivalent to $A_{\alpha,\beta}$, or otherwise, show that a full description of the solutions of $\mathcal{LS}(A_{\alpha,\beta},\, \mathbf{0}_3)$ is given by ($\sharp$):—

($\sharp$) $\mathbf{t}$ *is a solution of* $\mathcal{LS}(A_{\alpha,\beta},\, \mathbf{0}_3)$ *if and only if* $\mathbf{t}$ *is a scalar multiple of* $\mathbf{s}_{\alpha,\beta}$.

(c) i. Find a reduced row-echelon form which is row-equivalent to $A_{2,\beta}$ and whose $(4,1)$-th and $(4,2)$-th entries do not depend on β.

ii. Does $\mathcal{LS}(A_{2,1},\, \mathbf{0}_3)$ has any non-trivial solution? Justify your answer.
If yes, also give a full description of its solutions.

iii. Suppose $\beta \neq 1$.
Does $\mathcal{LS}(A_{2,\beta},\, \mathbf{0}_3)$ has any non-trivial solution? Justify your answer.
If yes, give a full description of its solutions.

9. (a) Prove the statements below, with direct reference to the definition for the notion of *equivalent systems of linear equations*:—

i. *Suppose A is an $(m \times n)$-matrix, and $\mathbf{b}$ is a column vector with m entries. Then $\mathcal{LS}(A,\, \mathbf{b})$ is equivalent to $\mathcal{LS}(A,\, \mathbf{b})$.*

ii. *Let A, C be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{d}$ be column vectors with m entries. Suppose $\mathcal{LS}(A,\, \mathbf{b})$ is equivalent to $\mathcal{LS}(C,\, \mathbf{d})$. Then $\mathcal{LS}(C,\, \mathbf{d})$ is equivalent to $\mathcal{LS}(A,\, \mathbf{b})$.*

iii. *Let A, C, E be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{d}, \mathbf{f}$ be column vectors with m entries. Suppose $\mathcal{LS}(A,\, \mathbf{b})$ is equivalent to $\mathcal{LS}(C,\, \mathbf{d})$, and also suppose $\mathcal{LS}(C,\, \mathbf{d})$ is equivalent to $\mathcal{LS}(E,\, \mathbf{f})$. Then $\mathcal{LS}(A,\, \mathbf{b})$ is equivalent to $\mathcal{LS}(E,\, \mathbf{f})$.*

Remark. Because of what we have proved here, it makes sense to write

'*$\mathcal{LS}(A,\, \mathbf{b})$ is equivalent (as a system) to $\mathcal{LS}(A',\, \mathbf{b}')$*'

as

'*$\mathcal{LS}(A,\, \mathbf{b})$, $\mathcal{LS}(A',\, \mathbf{b}')$ are equivalent (to each other as systems)*'.

(b) Note that a result that we have established can be re-formulated as below, which is labelled Statement (P):—

(P) *Let A, A' be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{b}'$ be column vectors with m entries.*
Suppose $[\ A \mid \mathbf{b}\ ]$, $[\ A' \mid \mathbf{b}'\ ]$ are are row-equivalent.
Then $\mathcal{LS}(A,\, \mathbf{b})$, $\mathcal{LS}(A',\, \mathbf{b}')$ are equivalent (as systems).

Interchanging the 'relative positions' of the 'assumption' and 'conclusion' in Statement (P), we formally write down a statement, labelled Statement (Q):

(Q) *Let A, A' be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{b}'$ be column vectors with m entries.*
Suppose $\mathcal{LS}(A, \mathbf{b})$, $\mathcal{LS}(A', \mathbf{b}')$ are equivalent (as systems).
Then $[\ A \mid \mathbf{b}\ ]$, $[\ A' \mid \mathbf{b}'\ ]$ are row-equivalent.

(Statement (Q) is called the **converse** of Statement (P).)

Show that the (Q) is false by writing down a counter-example against (Q) (and justify your answer).

Hint. Think of inconsistent systems with two linear equations and two unknonws, in which there are amongst the 'givens' as many 0's as possible.

(c) Inserting the word '*not*' into both the 'assumption' and 'conclusion' at appropriate places, we formally write down a statement, labelled Statement (R):

(R) *Let A, A' be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{b}'$ be column vectors with m entries.*
Suppose $\mathcal{LS}(A, \mathbf{b})$, $\mathcal{LS}(A', \mathbf{b}')$ are not equivalent (as systems).
Then $[\ A \mid \mathbf{b}\ ]$, $[\ A' \mid \mathbf{b}'\ ]$ are not row-equivalent.

(Statement (R) is called the **contrapositive (re-formulation)** of Statement (P).)

You may take for granted that Statement (R) is true; in fact, from the point of view of logic, Statement (R) and Statement (P) are the same mathematical statement.

Apply Statement (R) to show that the matrices labelled C, D in each part below are not row-equivalent:—

i. $C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

ii. $C = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 2 & 7 & 4 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 8 & 5 \end{bmatrix}$.

iii. $C = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 & -1 & 3 \\ 2 & -4 & 1 & 1 & 2 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 5 & 5 \end{bmatrix}$.

Remark. In each part, read each of C, D as the augmented matrix of some system. Then spot (or find) a column vector which is a solution of one of the system but which is not a solution of the other system.