2.3.2. Exercise: Existence of reduced row-echelon form row-equivalent to given matrix, (and the uniqueness question).

1. (a) Prove the statement ($\sharp$):

   ($\sharp$) *Let A, B be $(p \times q)$-matrices. Suppose B is resultant from the application of some row-operation ρ on A. Then $[\ B \mid M[\rho]\ ] = M[\rho][\ A \mid I_p\ ]$.*

   (b) Prove the statement ($\natural$):

   ($\natural$) *Suppose A is a $(p \times q)$-matrix. Then there exist some $(p \times q)$-matrices $A^\sharp, A'$ and some $(p \times p)$-matrices G, H such that the statements $(1), (2), (3), (4)$ hold simultaneously:*

   (1) *$A^\sharp$ is a row-echelon form.*

   (2) *A' is a reduced row-echelon form.*

   (3) *G, H are products of some row-operation matrices.*

   (4) *The equalities $[\ A^\sharp \mid G\ ] = G[\ A \mid I_p\ ]$, $[\ A' \mid H\ ] = H[\ A \mid I_p\ ]$ hold.*

2. For each part below, consider the given matrix, which is denoted by C here.

   - Apply Gaussian elimination to obtain some reduced row-echelon form C' which is row-equivalent to C.
   - Obtain some equality of the form $C' = HC$.

   (a) $C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$.

   (b) $C = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{bmatrix}$.

   (c) $C = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{bmatrix}$.

   (d) $C = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$.

   (e) $C = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{bmatrix}$.

   (f) $C = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix}$.

3. Let $A, A^\sharp, A'$ be $(p \times q)$-matrices.

   Suppose

   - $A^\sharp$ is a row-echelon form which is row-equivalent to A under some sequence of row operations

   $$A \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_m} A^\sharp$$

   - A' is a reduced row-echelon form which is row-equivalent to $A^\sharp$ under some sequence of row operations

   $$A^\sharp \xrightarrow{\sigma_1} \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} A'$$

   Write $J = M[\rho_m] \cdots M[\rho_2]M[\rho_1]$ and $K = M[\sigma_n] \cdots M[\sigma_2]M[\sigma_1]$. (Note that $A^\sharp = JA$ and $A' = KA^\sharp$.)

   Suppose the rank of $A^\sharp$, and that of A', are both p. (So all rows of $A^\sharp$ are non-zero rows, and all rows of A' are non-zero rows.)

   (a) Let B be the $(p \times 2q)$-matrix given by $B = [\ A \mid A\ ]$.

   Give, in terms of $A^\sharp, A'$, some row-echelon form $B^\sharp$ and some reduced row-echelon form B' which are row-equivalent to B.

   Give some appropriate equalities which relate $B, B^\sharp, B'$, and J, K.

(b) Let C be the $(2p \times q)$-matrix given by $C = \left[\begin{array}{c} A \\ \hline A \end{array}\right]$.

Give, in terms of $A^\sharp, A'$, some row-echelon form $C^\sharp$ and some reduced row-echelon form C' which are row-equivalent to C.

Give some appropriate equalities which relate $C, C^\sharp, C'$, and J, K.

(c) Let D be the $(2p \times 2q)$-matrix given by $D = \left[\begin{array}{c|c} A & \mathcal{O}_{p\times q} \\ \hline \mathcal{O}_{p\times q} & A \end{array}\right]$.

Give, in terms of $A^\sharp, A'$, some row-echelon form $D^\sharp$ and some reduced row-echelon form D' which are row-equivalent to D.

Give some appropriate equalities which relate $D, D^\sharp, D'$, and J, K.

(d) Let E be the $(2p \times 2q)$-matrix given by $E = \left[\begin{array}{c|c} A & A \\ \hline \mathcal{O}_{p\times q} & A \end{array}\right]$.

Give, in terms of $A^\sharp, A'$, some row-echelon form $E^\sharp$ and some reduced row-echelon form E' which are row-equivalent to E.

Give some appropriate equalities which relate $E, E^\sharp, E'$, and J, K.