2.3.1 Proof of existence of row-echelon form and reduced row-echelon form row-equivalent to given matrix.

0. The material in this appendix is supplementary.

1. We are going to prove Theorem (1) and Theorem (3).

   Theorem (1). (Existence of row-echelon form which is row-equivalent to a given matrix.)

   Suppose that C is a matrix.

   Then there exists some row-echelon form $C^\sharp$ such that $C^\sharp$ is row-equivalent to C.

   Theorem (3). (Existence of reduced row-echelon form which is row-equivalent to a given matrix.)

   Suppose that C is a matrix.

   Then there exists some reduced row-echelon form C' such that C' is row-equivalent to C.

2. **Proof of Theorem (1).**

   The result follows once we verify the statement below (with the help of mathematical induction):—

   For any positive integer m, if A is a matrix with m rows, then there is some row-echelon form $A^\sharp$ with m rows such that $A^\sharp$ is row-equivalent to A.

   Denote by $P(m)$ the proposition below:—

   If A is a matrix with m rows, then there is some row-echelon form $A^\sharp$ with m rows such that $A^\sharp$ is row-equivalent to A.

   (a) Suppose A is a matrix with 1 row. Then A itself is a row-echelon form.
   Hence $P(1)$ is true.

   (b) Suppose $P(k)$ is true. We verify that $P(k+1)$ is true:—
   Suppose A is a matrix with $k+1$ rows.
   If A the zero matrix, then A is a row-echelon form.
   From now on suppose A is not the zero matrix.

      i. There is a left-most non-zero column in A, say, the j-th column.
      In this column, there is a top-most non-zero entry, say, the i-th entry, of value β.
      - If $i = 1$, take $A_1 = A$.
      - If $i \neq 1$, then apply the row operation $R_1 \leftrightarrow R_i$ to obtain some the matrix A_1:

      $$A \xrightarrow{R_1 \leftrightarrow R_i} A_1.$$

      As in A, all entries in A_1 strictly to the left of the j-th column are 0.

      ii. Denote the respective entries of the j-th column of A_1, from the second entry downwards, by $\alpha_2, \alpha_3, \cdots, \alpha_{k+1}$.
      Apply on A_1 the sequence of row operations to obtain the matrix A_2:

      $$A_1 \xrightarrow{-\frac{\alpha_2}{\beta}R_1 + R_2} \xrightarrow{-\frac{\alpha_3}{\beta}R_1 + R_3} \cdots \xrightarrow{-\frac{\alpha_{k+1}}{\beta}R_1 + R_{k+1}} A_2.$$

      Note that all entries of A to the left of the j-th column are 0. Then all entries strictly to the left of the j-th column in each of the matrices resultant from the application of the row operations above remain to be 0.
      By construction, every entry in the j-th column from the second entry downward are all 0.
      Therefore

      $$A_2 = \left[\begin{array}{c|c|c} \mathcal{O}_{1\times(j-1)} & \beta & \mathbf{u} \\ \hline \mathcal{O}_{k\times(j-1)} & \mathbf{0}_k & V \end{array}\right]$$

      in which $\mathbf{u}$ is some row vector, and V is some matrix with k rows.

      iii. By $P(k)$, the matrix V is row-equivalent to some row-echelon form V', with some sequence of row operations joining V to V':

      $$V \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_\ell} V'.$$

      iv. $\rho_1, \rho_2, \cdots, \rho_\ell$ are row operations on matrices with k rows define.
      By making a change of each row index involved in $\rho_1, \rho_2, \cdots, \rho_\ell$ by an increase of 1, we define the respective row operations $\widehat{\rho_1}, \widehat{\rho_2}, \cdots, \widehat{\rho_\ell}$ on matrices with $k+1$ rows. By definition, each of these row operations leaves the first row of any matrix unchanged.

v. An application on A_2 of the sequence of row operations $\widehat{\rho_1}, \widehat{\rho_2}, \cdots, \widehat{\rho_\ell}$ results in the matrix A':

$$A_2 = \left[\begin{array}{c|c|c} \mathcal{O}_{1\times(j-1)} & \beta & \mathbf{u} \\ \hline \mathcal{O}_{k\times(j-1)} & \mathbf{0}_k & V \end{array}\right] \xrightarrow{\widehat{\rho_1}} \xrightarrow{\widehat{\rho_2}} \cdots \xrightarrow{\widehat{\rho_\ell}} A' = \left[\begin{array}{c|c|c} \mathcal{O}_{1\times(j-1)} & \beta & \mathbf{u} \\ \hline \mathcal{O}_{k\times(j-1)} & \mathbf{0}_k & V' \end{array}\right]$$

By construction, A' is a row-echelon form.

It follows that $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(m)$ is true for each positive integer m.

3. We proceed to prove Theorem (3). The hard work in the argument for Theorem (3) is done in proving Lemma (5).

Lemma (5). (Preparation for the proof of Theorem (3).)

For any natural number r, if A is a row-echelon form of rank r, there exists some reduced row-echelon form A' of rank r such that A' is row-equivalent to A.

4. **Proof of Lemma (5).**

[We apply the method of mathematical induction.]

Denote by $P(r)$ the proposition below:—

If A is a row-echelon form of rank r, there exists some reduced row-echelon form A' of rank r such that A' is row-equivalent to A.

(a) We verify that $P(0)$ is true:—

Suppose A is a row-echelon form of rank 0. By assumption, there is no non-zero row in A. Then A is itself a reduced row echelon form of rank 0 which is row-equivalent to A.

(b) Suppose $P(k)$ is true. We verify that $P(k+1)$ is true:—

Suppose A is a row-echelon form of rank $k+1$, with pivot columns, from left to right, being the d_1-th, d_2-th, ..., d_k-th, d_{k+1}-th columns of A.

i. Write $A = [\ U \mid V\ ]$, in which:—
   - U consists of the columns of A strictly to the left of the d_{k+1}-th column of A, and
   - V consists of the columns of A to the right of d_{k+1}-th column of A, including the d_{k+1}-th column.

ii. By construction, U is a row-echelon form of rank k. Then, by $P(k)$, there exists some reduced-row echelon form U_* such that U_* is row-equivalent to U.

iii. By definition, U_* is resultant from the application on U of some sequence of row operations $\rho_1, \rho_2, \cdots, \rho_\ell$:

$$U \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_\ell} U_*.$$

Since the rows below the respective r-th rows of U, U_* are all 0, we assume, without loss of generality, that the application of any of $\rho_1, \rho_2, \cdots, \rho_\ell$ on a matrix affects the its top r rows.

iv. Applying on the whole of A the same sequence of row operations

$$A = [\ U \mid V\ ] \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_\ell} A_* = [\ U_* \mid V_*\ ],$$

we obtain some matrix A_* in which:—
   - V_* is some matrix consisting of the columns of A_* to the right of d_{k+1}-th column of A_*, including the d_{k+1}-th column.

v. Since the application of any of $\rho_1, \rho_2, \cdots, \rho_\ell$ on a matrix affects the its top r rows, A_* is a row-echelon form of rank $k+1$, with pivot columns (from left to right) being the d_1-th, d_2-th, ..., d_k-th, d_{k+1}-th columns of A_*.

vi. For each j, denote the j-th entry of the d_{k+1}-th column of A_* by α_j.
As the d_{k+1}-th column of A_* is a pivot column of A_*, we have $\alpha_{k+1} \neq 0$, and for each $j = k+2, k+3, \cdots$, $\alpha_j = 0$.

vii. Applying on A_* (and U_*, V_*) the (same) sequence of row operations below, we obtain A_{**} (and U_{**}, V_{**} respectively):

$$A_* = [\ U_* \mid V_*\ ] \xrightarrow{{\alpha_{k+1}}^{-1}R_{k+1}} \xrightarrow{-\alpha_1 R_{k+1}+R_1} \xrightarrow{-\alpha_2 R_{k+1}+R_2} \cdots \xrightarrow{-\alpha_k R_{k+1}+R_k} A_{**} = [\ U_{**} \mid V_{**}\ ]$$

viii. Since all entries to the left of the $(k+1)$-th entry of A_{**} are 0, all entries in the first $d_{k+1}-1$ columns in each iteration in this sequence remain unchanged throughout. Hence $U_{**} = U_*$.
By definition, the row operations in this sequence does not affect any row below the $k+1$-th row of any matrix. Hence all entries of V_{**} below the $(k+1)$-th row are 0.
Then A_{**} is a row-echelon form of rank $k+1$, with pivot columns (from left to right) being the d_1-th, d_2-th, ..., d_k-th, d_{k+1}-th columns of A_*.

ix. Since U_* is a reduced row-echelon form, for each $j = 1, 2, \cdots, k$, the d_j-th column of A_{**} has exactly one non-zero entry, namely, the j-th entry, whose value is 1.
By construction, the d_{k+1}-th column of A_{**} has exactly one non-zero entry, namely, the $(k+1)$-th entry, whose value is 1.
Then A_{**} is a reduced row-echelon form. By construction, A_{**} is row-equivalent to A.

It follows that $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(r)$ is true for any natural number r.

5. In the argument for Lemma (5), we have in fact proved more than what is stated in Lemma (5). This is put into Theorem (6).

Theorem (6).

Suppose A is a row-echelon form of rank r, whose pivot columns are, say, the d_1-th, d_2-th, ..., d_r-th columns of A.

Then there is some reduced row-echelon form A' of rank r, whose pivot columns are the d_1-th, d_2-th, ..., d_r-th columns of A', so that A' is row-equivalent to A.

6. We now complete the argument for Theorem (3).

Proof of Theorem (3).

Suppose that C is a matrix.

By Theorem (1), there exists some row-echelon form $C^\sharp$ such that $C^\sharp$ is row-equivalent to C.

By Lemma (5), there exists some reduced row-echelon form C' such that C' is row-equivalent to $C^\sharp$.

It follows that C' is a reduced row-echelon form which is row-equivalent to C.