2.2.2 Answers to Exercise.

1. (a) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\mathcal{LS}(A, \mathbf{b})$ reads: $\left\{ \begin{array}{rcrcrcr} x_1 & & & + & 2x_3 & = & 3 \\ & & x_2 & - & 3x_3 & = & 4 \end{array} \right.$

$\mathcal{LS}(A, \mathbf{b})$ is consistent.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if there exists some number u such that $\mathbf{t} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + u \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$.

$\mathcal{LS}(A, \mathbf{b})$ reads: $\left\{ \begin{array}{rcrcrcrcr} x_1 & & & & & & & = & 3 \\ & & x_2 & - & 2x_3 & & & = & 4 \\ & & & & & & x_4 & = & -1 \end{array} \right.$

$\mathcal{LS}(A, \mathbf{b})$ is consistent.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if there exists some number u such that $\mathbf{t} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \end{bmatrix} + u \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$\mathcal{LS}(A, \mathbf{b})$ reads: $\left\{ \begin{array}{rcrcrcrcr} x_1 & & & + & 2x_3 & & & = & 0 \\ & & x_2 & + & 5x_3 & & & = & 0 \\ & & & & & & x_4 & = & 0 \\ & & & & & & 0 & = & 1 \end{array} \right.$

$\mathcal{LS}(A, \mathbf{b})$ is inconsistent.

(d) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \\ 4 \\ 0 \end{bmatrix}$.

$\mathcal{LS}(A, \mathbf{b})$ reads: $\left\{ \begin{array}{ccccccr} x_1 & & & & & = & -1 \\ & x_2 & & & & = & -2 \\ & & x_3 & & & = & 2 \\ & & & x_4 & & = & 3 \\ & & & & x_5 & = & 4 \\ & & & & & = & 0 \end{array} \right.$

$\mathcal{LS}(A, \mathbf{b})$ is consistent.

The only solution of $\mathcal{LS}(A, \mathbf{b})$ is given by $\begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 1 & 2 & 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ -1 \\ 0 \end{bmatrix}$.

$\mathcal{LS}(A, \mathbf{b})$ reads: $\left\{ \begin{array}{rcrcrcrcrcrcr} x_1 & + & 2x_2 & & & & & - & 2x_5 & + & 3x_6 & = & 4 \\ & & & & x_3 & & & & & - & 2x_6 & = & 3 \\ & & & & & & x_4 & - & 3x_5 & + & 5x_6 & = & -1 \\ & & & & & & & & & & 0 & = & 0 \end{array} \right.$

$\mathcal{LS}(A, \mathbf{b})$ is consistent.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if there exist some numbers u, v, w such that $\mathbf{t} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} +$

$$v \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -3 \\ 0 \\ 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

(f) $A = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 & 0 & -1 & 4 \\ 0 & 1 & -3 & 0 & 4 & 0 & -5 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

$\mathcal{LS}(A,\, \mathbf{b})$ reads: $\begin{cases} x_1 + 2x_3 - 2x_5 - x_7 + 4x_8 = 3 \\ x_2 - 3x_3 + 4x_5 - 5x_7 - x_8 = -2 \\ x_4 + x_5 - 3x_7 + 2x_8 = -4 \\ x_6 + 5x_7 + 3x_8 = 2 \\ 0 = 0 \\ 0 = 0 \end{cases}$

$\mathcal{LS}(A,\, \mathbf{b})$ is consistent.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{b})$ if and only if there exist some numbers u_1, u_2, u_3, u_4 such that $\mathbf{t} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} +$

$$u_2 \begin{bmatrix} 2 \\ -4 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ 5 \\ 0 \\ 3 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} -4 \\ 1 \\ 0 \\ -2 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

(g) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\mathcal{LS}(A,\, \mathbf{b})$ reads: $\begin{cases} x_1 = 3 \\ x_2 = 5 \\ x_3 = 7 \\ \quad = 0 \\ \quad = 0 \\ \quad = 0 \end{cases}$

$\mathcal{LS}(A,\, \mathbf{b})$ is consistent.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{b})$ if and only if there exist some number u such that $\mathbf{t} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

2. (a) $\mathcal{LS}(A,\, \mathbf{0}_2)$ reads: $\begin{cases} x_1 + 2x_3 + 3x_4 = 0 \\ x_2 - 3x_3 + 4x_4 = 0 \end{cases}$

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{0}_2)$ if and only if there exist some number u, v such that $\mathbf{t} = u \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \end{bmatrix}$.

$\mathcal{LS}(A,\, \mathbf{0}_2)$ has some non-trivial solution.

(b) $\mathcal{LS}(A,\, \mathbf{0}_3)$ reads: $\begin{cases} x_1 + 3x_5 = 0 \\ x_2 - 2x_3 + 4x_5 = 0 \\ x_4 - x_5 = 0 \end{cases}$

$\mathcal{LS}(A,\, \mathbf{0}_3)$ has some non-trivial solution.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{0}_3)$ if and only if there exist some number u, v such that $\mathbf{t} = u \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

(c) $\mathcal{LS}(A,\,\mathbf{0}_8)$ reads: $\left\{\begin{array}{rrrrrcl} x_1 & & & & & = & 0 \\ & x_2 & & & & = & 0 \\ & & x_3 & & & = & 0 \\ & & & x_4 & & = & 0 \\ & & & & x_5 & = & 0 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \end{array}\right.$

$\mathcal{LS}(A,\,\mathbf{0}_8)$ has no non-trivial solution.

(d) $\mathcal{LS}(A,\,\mathbf{0}_4)$ reads: $\left\{\begin{array}{rcrrrcrcrcrcl} x_1 & + & 2x_2 & & & - & 2x_5 & + & 3x_6 & + & 4x_7 & = & 0 \\ & & & x_3 & & & & - & 2x_6 & + & 3x_7 & = & 0 \\ & & & & x_4 & - & 3x_5 & + & 5x_6 & - & x_7 & = & 0 \\ & & & & & & & & & & 0 & = & 0 \end{array}\right.$

$\mathcal{LS}(A,\,\mathbf{0}_4)$ has some non-trivial solution.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\,\mathbf{0}_3)$ if and only if there exist some number u_1, u_2, u_3, u_4 such that $\mathbf{t} = u_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} -3 \\ 0 \\ 2 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(e) $\mathcal{LS}(A,\,\mathbf{0}_6)$ reads: $\left\{\begin{array}{rrcrrcrrcrcrcrcl} x_1 & & + & 2x_3 & & - & 2x_5 & & - & x_7 & + & 4x_8 & + & 3x_9 & = & 0 \\ & x_2 & - & 3x_3 & & + & 4x_5 & & - & 5x_7 & - & x_8 & - & 2x_9 & = & 0 \\ & & & & x_4 & + & x_5 & & - & 3x_7 & + & 2x_8 & - & 4x_9 & = & 0 \\ & & & & & & & x_6 & + & 5x_7 & + & 3x_8 & + & 2x_9 & = & 0 \\ & & & & & & & & & & & & & 0 & = & 0 \\ & & & & & & & & & & & & & 0 & = & 0 \end{array}\right.$

$\mathcal{LS}(A,\,\mathbf{0}_6)$ has some non-trivial solution.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\,\mathbf{0}_3)$ if and only if there exist some number u_1, u_2, u_3, u_4, u_5 such that $\mathbf{t} = u_1 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 2 \\ -4 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ 5 \\ 0 \\ 3 \\ 0 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} -4 \\ 1 \\ 0 \\ -2 \\ 0-3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u_5 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 4 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(f) $\mathcal{LS}(A,\,\mathbf{0}_6)$ reads: $\left\{\begin{array}{rrrrcl} x_1 & & & & = & 0 \\ & x_2 & & & = & 0 \\ & & x_4 & & = & 0 \\ & & & x_5 & = & 0 \\ & & & 0 & = & 0 \\ & & & 0 & = & 0 \end{array}\right.$

$\mathcal{LS}(A,\,\mathbf{0}_6)$ has some non-trivial solution.

$\mathbf{t}$ is a solution of $\mathcal{LS}(A,\,\mathbf{0}_6)$ if and only if there exists some number u such that $\mathbf{t} = u \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

3. (a) i. 1-st, 2-nd, 4-th, 6-th columns.
   ii. *Comment.*

Note that the j-th entry of the j-th pivot column from left to right in C is 1, and every other entry of such a pivot column is 0. This is the key to the issue of linear independence.

iii. *Comment.*
Note that whenever $i > \ell$, the first non-zero entry in the i-th non-zero row in C is strictly to the left of the first non-zero entry in the ℓ-th non-zero row in C. This is the key to the issue of linear independence.

iv. *Comment.*
Each free column in C is a linear combination of the pivot columns in C strictly to its left.

v. ——

(b)

4. (a) True.

(b) True.

(c) True.

(d) True.

(e) True.

(f) False.