2.2.2 Exercise: Row-echelon forms and reduced row-echelon forms.

1. For each part below, consider the reduced row-echelone form denoted by C here. Such a C is supposed to be the augmented matrix representation of some system of linear equations $\mathcal{LS}(A,\, \mathbf{b})$, in which A is some $(m \times n)$-matrix.
   - Identify its coefficient matrix A and vector of constants $\mathbf{b}$.
   - Present the system $\mathcal{LS}(A,\, \mathbf{b})$ explicitly, displaying all m individual equations in the system, and the n unknowns $x_1, x_2, \cdots, x_n$.
   - Determine whether the system $\mathcal{LS}(A,\, \mathbf{b})$ is consistent.
   - Where the system $\mathcal{LS}(A,\, \mathbf{b})$ is consistent, give a full description of all solutions of $\mathcal{LS}(A,\, \mathbf{b})$.

   (a) $C = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & 4 \end{array}\right]$.

   (b) $C = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$.

   (c) $C = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$.

   (d) $C = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

   (e) $C = \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & -2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 & -3 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

   (f) $C = \left[\begin{array}{cccccccc|c} 1 & 0 & 2 & 0 & -2 & 0 & -1 & 4 & 3 \\ 0 & 1 & -3 & 0 & 4 & 0 & -5 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -3 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

   (g) $C = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

2. For each part below, consider the $(m \times n)$-reduced row-echelon form denoted by A here. Such an A is supposed to be the coefficient matrix of some homogeneous system $\mathcal{LS}(A,\, \mathbf{0}_m)$.
   - Present the system $\mathcal{LS}(A,\, \mathbf{0}_m)$ explicitly, displaying all m individual equations in the system, and the n unknowns $x_1, x_2, \cdots, x_n$.
   - Determine whether the system $\mathcal{LS}(A,\, \mathbf{b})$ has any non-trivial solutions.
   - Where the system $\mathcal{LS}(A,\, \mathbf{b})$ has some non-trivial solution, give a full description of all solutions of $\mathcal{LS}(A,\, \mathbf{b})$.

   (a) $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & 4 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} 1 & 2 & 0 & 0 & -2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 & -3 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (e) $A = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 & 0 & -1 & 4 & 3 \\ 0 & 1 & -3 & 0 & 4 & 0 & -5 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -3 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (f) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

3. *The two parts of this question is un-related to each other, though the hidden agenda in part* (a) *is to provide you some feeling on how you may approach part* (b).

   (a) Let $C = \begin{bmatrix} 1 & 0 & -3 & 0 & 2 & 0 & -1 \\ 0 & 1 & -4 & 0 & 5 & 0 & 6 \\ 0 & 0 & 0 & 1 & 8 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   Note that C is a reduced row-echelon form.
   Denote the columns of C, from left to right, by $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6, \mathbf{c}_7$, and denote the rows of C, from top to bottom, by $\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3, \hat{\mathbf{c}}_4$.

      i. Name the pivot columns of C.

      ii. Verify that the pivot columns of C are linearly independent over the reals.

iii. Verify that the non-zero rows of C are linearly independent over the reals.
iv. Verify that each free column is a linear combination of the pivot columns strictly to its left over the reals.
v. Verify that every linear combination of the columns of C over the reals is a linear combination of the pivot columns of C over the reals.

(b) Let B be a (9×63)-matrix. Suppose B is a reduced row-echelon form.

Denote the columns of B, from left to right, by $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_{63}$, and denote the rows of B, from top to bottom, by $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \cdots, \hat{\mathbf{b}}_9$.

Suppose that the pivot columns of B are the 1-st, 2-nd, 4-th, 8-th, 16-th, 32-nd columns of B.

i. Describe the column vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4, \mathbf{b}_8, \mathbf{b}_{16}, \mathbf{b}_{32}$, by explicitly giving their entries.
ii. Show that the pivot columns of B are linearly independent over the reals.
iii. Name the non-zero rows of B.
iv. Show that the non-zero rows of B are linearly independent over the reals.
v. Show that $\mathbf{b}_{15}$ is a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4, \mathbf{b}_8$ over the reals.
vi. Show that each linear combination of the columns of B over the reals is a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4, \mathbf{b}_8, \mathbf{b}_{16}, \mathbf{b}_{32}$ over the reals.

4. For each of the statements below, determine whether it is true or false. If it is true, provide a proof. If it is false, provide a counter-example against it.

(a) *Let C be an $(n \times n)$-square matrix. Suppose C is a reduced-row echelon form, and every column of C is a pivot column. Then $C = I_n$.*

(b) *Let C be an $(n \times n)$-square matrix. Suppose C is a reduced-row echelon form, and every row of C is a non-zero row. Then $C = I_n$.*

(c) *Let C be an $(m \times n)$-matrix. Suppose C is a reduced-row echelon form, and every column of C is a pivot column. Then $m \geq n$.*

(d) *Let C be an $(m \times n)$-matrix. Suppose C is a reduced-row echelon form, and every row of C is a non-zero row. Then $m \leq n$.*

(e) *Let C be an $(m \times n)$-matrix. Suppose C is a reduced-row echelon form, and every column of C is a pivot column. Then $C = \left[\begin{array}{c} I_n \\ \hline \mathcal{O} \end{array}\right]$.*

(f) *Let C be an $(m \times n)$-matrix. Suppose C is a reduced-row echelon form, and every row of C is a non-zero. Then $C = [\ I_m \mid \mathcal{O}\ ]$.*