2.2.1 Appendix: Reduced row-echelon forms of small sizes.

The material in this appendix is supplementary.

We describe the various types of reduced row-echelon forms in $(p \times q)$ -matrices for 'small' p, q. The symbols a, b, c, d, when they are present, stand for various choices of number. Each distinct choice gives rise to a distinct reduced row-echelon form.

1. (1×1) -matrices:—

2. (1×2) -matrices:—

$$[0 \ 0]; [1 \ a], [0 \ 1].$$

3. (1×3) -matrices:—

$$[0 \ 0 \ 0]; [1 \ a \ b], [0 \ 1 \ a], [0 \ 0 \ 1]$$

4. (1×4) -matrices:—

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & a & b & c \end{bmatrix}, \begin{bmatrix} 0 & 1 & a & b \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & a \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. (2×1) -matrices:—

 $\begin{bmatrix} 0\\0 \end{bmatrix}; \begin{bmatrix} 1\\0 \end{bmatrix}.$

6. (2×2) -matrices:—

 $\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]; \qquad \left[\begin{array}{cc} 1 & a \\ 0 & 0 \end{array}\right], \qquad \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]; \qquad \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$

7. (2×3) -matrices:—

 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}, \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

8. (2×4) -matrices:—

 $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & a & b & c \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & a & b \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & a & 0 & b \\ 0 & 1 & c & d \end{bmatrix}, \begin{bmatrix} 1 & a & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}, \begin{bmatrix} 1 & a & b & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

9. (3×1) -matrices:—

$\left[\begin{array}{c}0\\0\\0\end{array}\right];\qquad \left[\begin{array}{c}1\\0\\0\end{array}\right].$

10. (3×2) -matrices:—

11. (3×3) -matrices:—

 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & a \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$

12. (3×4) -matrices:—

$\left[\begin{array}{ccc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$ $\left[\begin{array}{ccc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right]$ $\left[\begin{array}{ccc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right]$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};$ $\begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix},$ $\begin{bmatrix} 0 & a \\ 1 & c \end{bmatrix},$	$\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ b \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{array}{cccccccc} 0 & 1 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \end{array}$	$\begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ \end{bmatrix}$	$\left[\begin{array}{cccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{cccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$	$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ a \\ b \\ 0 \end{bmatrix}, \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$	$\begin{array}{ccc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \end{array}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};$ $\begin{bmatrix} a & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$	$\left[\begin{array}{rrrrr} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right];$
13. (4×4) -matrices:—										
$\left[\begin{array}{ccc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$	$\left[\begin{array}{ccc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ c & d \\ 0 & 0 \\ 0 & 0 \\ 0 & b \\ 1 & c \\ 0 & 0 \\ \end{array} \right],$	$\left[\begin{array}{ccccc} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$	$\begin{bmatrix} c \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ $	$ \begin{smallmatrix} 0 & 1 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & 0 \\ 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \end{smallmatrix} $	$\begin{bmatrix} b \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0$	$\left[\begin{array}{cccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 &$	$\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\\begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}, \\\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \\\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 1 \end{bmatrix}; \\\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 1 \end{bmatrix}; \\\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \\\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\left[\begin{array}{ccc} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ a & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \end{array} \right],$	$\left[\begin{array}{rrrrr} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];$