2.1.1 Answers to Exercise.

1. (a) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$.

The matrix presentation of (S) reads: $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$.

The augmented matrix representation of (S) reads: $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right]$.

The vector presentation of (S) reads: $x_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ -7 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$.

t is a solution.
u is not a solution.

(b) $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

The matrix presentation of (S) reads: $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

The augmented matrix representation of (S) reads: $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{array}\right]$.

The vector presentation of (S) reads: $x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

t is a solution.
u is not a solution.

(c) $A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ -2 & -1 & -3 & 3 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

The matrix presentation of (S) reads: $\begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ -2 & -1 & -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

The augmented matrix representation of (S) reads: $\left[\begin{array}{ccccc|c} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{array}\right]$.

The vector presentation of (S) reads: $x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

t is a solution.
u is a solution.

(d) $A = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 \\ -1 & 2 & 1 & -1 & 0 & -2 \\ 2 & -4 & -1 & 3 & 2 & 1 \\ 3 & -6 & -1 & 5 & 4 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}$.

The matrix presentation of S reads: $\begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 \\ -1 & 2 & 1 & -1 & 0 & -2 \\ 2 & -4 & -1 & 3 & 2 & 1 \\ 3 & -6 & -1 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}$.

The augmented matrix representation of (S) reads: $\left[\begin{array}{cccccc|c} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{array}\right]$.

The vector presentation of (S) reads:

$$x_1 \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ -4 \\ -6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 3 \\ 5 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ 2 \\ 4 \end{bmatrix} + x_6 \begin{bmatrix} -7 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}.$$

t is a solution.
u is a solution.

2. (a) $A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & 3 & 2 \\ 1 & -5 & 0 & 4 \end{bmatrix}$

   Matrix presentation of (H) reads: $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & 3 & 2 \\ 1 & -5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

   $\mathbf{t}$ is a non-trivial solution of $\mathcal{LS}(A, \mathbf{0}_3)$.

   $\mathbf{u}$ is not a non-trivial solution of $\mathcal{LS}(A, \mathbf{0}_3)$.

   (b) $A = \begin{bmatrix} 1 & -2 & 4 & 5 & 2 \\ -2 & 5 & -6 & -11 & -8 \\ 3 & -5 & 14 & 15 & 3 \\ 2 & -5 & 6 & 15 & 12 \end{bmatrix}$

   Matrix presentation of (H) reads: $\begin{bmatrix} 1 & -2 & 4 & 5 & 2 \\ -2 & 5 & -6 & -11 & -8 \\ 3 & -5 & 14 & 15 & 3 \\ 2 & -5 & 6 & 15 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

   $\mathbf{t}$ is a non-trivial solution of $\mathcal{LS}(A, \mathbf{0}_4)$.

   $\mathbf{u}$ is not a non-trivial solution of $\mathcal{LS}(A, \mathbf{0}_4)$.

3. (a) *Comment.*

   When the 'cosmetics' in the language of systems of linear equations is stripped away, it is the 'algebraic identity' $A(\mathbf{u} + \mathbf{u}') = A\mathbf{u} + A\mathbf{u}' = \mathbf{b} + \mathbf{b}'$.

   (b) *Comment.*

   When the 'cosmetics' in the language of systems of linear equations is stripped away, it is the 'algebraic identity' $A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{b}$.

4. *Comment.*

   It is the pair of results described here that makes homogeneous systems special amongst all systems.

   The second result says that there is a 'linear structure' inside the collection of all solutions of an arbitrary homogeneous system, in the sense that linear combinations of solutions of such a system are always solutions of this system again.

5. (a) ——

   (b) One possible choice of counter-example is to take $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

6. (a) True.

   (b) False. A counter-example is obtained by placing the equation '$0 = 1$' beneath the other three equations in a concrete consistent system (T).

   (c) False. A counter-example is obtained by 'deleting' the terms involving 'x_6' from a concrete consistent (S) which, amongst its four equations, includes the equation '$x_6 = 1$'.

   (d) True.

7. (a) True.

   (b) False. A counter-example is obtained by choosing $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   (c) True.

   (d) False. A counter-example is obtained by choosing $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} 2 \end{bmatrix}$.

   (e) True.

   (f) True.

8. ——

9. (a) ——

   (b) The converse of $(\sharp)$ is true.

10. *Comment.*

    The key is the identity '$A(A\mathbf{b}) = \mathbf{b}$'.

11. (a) ——

    (b) The statement $(\natural)$ is false. A counter-example against $(\natural)$ is provided by $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.