2.1.1 Exercise: Systems of linear equations.

1. For each part below, consider the given system of linear equations, which is denoted by (S) here.
   - Display (S) in its respective matrix presentation. Identify its coefficient matrix A and vector of constants $\mathbf{b}$. Also give its augmented matrix representation and vector presentation.
   - For the given vectors $\mathbf{t}, \mathbf{u}$ in each part whether they are solutions of the system (S).

(a) $(S): \left\{\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & x_4 & = & 7 \\ x_1 & + & x_2 & + & x_3 & - & x_4 & = & 3 \\ 3x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 1 \end{array}\right.$.

$\mathbf{t} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}$.

(b) $(S): \left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array}\right.$.

$\mathbf{t} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

(c) $(S): \left\{\begin{array}{rcrcrcrcrcr} & & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 2 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & + & 3x_5 & = & 4 \\ -2x_1 & - & x_2 & - & 3x_3 & + & 3x_4 & + & x_5 & = & 3 \end{array}\right.$.

$\mathbf{t} = \begin{bmatrix} 8 \\ -9 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 10 \\ -9 \\ 1 \\ 6 \\ -1 \end{bmatrix}$.

(d) $(S): \left\{\begin{array}{rcrcrcrcrcrcr} & & & & 2x_3 & + & 3x_4 & + & 5x_5 & - & 7x_6 & = & 12 \\ -x_1 & + & 2x_2 & + & x_3 & - & x_4 & & & - & 2x_6 & = & 0 \\ 2x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & + & 2x_5 & + & x_6 & = & 5 \\ 3x_1 & - & 6x_2 & - & x_3 & + & 5x_4 & + & 4x_5 & & & = & 10 \end{array}\right.$.

$\mathbf{t} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \\ 2 \end{bmatrix}$.

2. For each part below, consider the given homogeneous system of equations, which is denoted by (H) here.
   - Display (H) in its respective matrix presentation. Identify its coefficient matrix A.
   - For the given vectors $\mathbf{t}, \mathbf{u}$ in each part whether they are non-trivial solutions of the system (H).

(a) $(H): \left\{\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & & & = & 0 \\ & & x_2 & + & 3x_3 & + & 2x_4 & = & 0 \\ x_1 & - & 5x_2 & & & + & 4x_4 & = & 0 \end{array}\right.$.

$\mathbf{t} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

(b) $(H): \left\{\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & 4x_3 & + & 5x_4 & + & 2x_5 & = & 0 \\ -2x_1 & + & 5x_2 & - & 6x_3 & - & 11x_4 & - & 8x_5 & = & 0 \\ 3x_1 & - & 5x_2 & + & 14x_3 & + & 15x_4 & + & 3x_5 & = & 0 \\ 2x_1 & - & 5x_2 & + & 6x_3 & + & 15x_4 & + & 12x_5 & = & 0 \end{array}\right.$.

$\mathbf{t} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

3. Let A be an $(m \times n)$-matrix, $\mathbf{b}, \mathbf{b}'$ be vectors with m entries, and c be a number.

   Prove the statements below, with direct reference to the appropriate definitions:—

(a) *Suppose* $\mathbf{t}$ *is a solution of* $\mathcal{LS}(A,\, \mathbf{b})$*, and* $\mathbf{t}'$ *is a solution of* $\mathcal{LS}(A,\, \mathbf{b}')$*. Then* $\mathbf{t}+\mathbf{t}'$ *is a solution of* $\mathcal{LS}(A,\, \mathbf{b}+\mathbf{b}')$*.*

(b) *Suppose* $\mathbf{u}$ *is a solution of* $\mathcal{LS}(A,\, \mathbf{b})$*. Then* $c\mathbf{u}$ *is a solution of* $\mathcal{LS}(A,\, c\mathbf{b})$*.*

4. Let A be an $(m \times n)$-matrix.

   Prove the statements below, with direct reference to the appropriate definitions:—

   (a) *The homogeneous system* $\mathcal{LS}(A,\, \mathbf{0}_m)$ *has a solution.*

   (b) *Suppose* $\mathbf{t}, \mathbf{t}'$ *are solutions of the homogeneous system* $\mathcal{LS}(A,\, \mathbf{0}_m)$*, and* c, c' *are numbers. Then* $c\mathbf{t} + c'\mathbf{t}'$ *is a solution of* $\mathcal{LS}(A,\, \mathbf{0}_m)$*.*

5. (a) Let A be an $(m \times n)$-matrix, and $\mathbf{b}$ be a column vector with m entries.

   Prove the statements below, with direct reference to the appropriate definitions:—

   i. *Suppose* $\mathcal{LS}(A,\, \mathbf{b})$ *has two distinct solutions. Then the homogeneous system* $\mathcal{LS}(A,\, \mathbf{0}_m)$ *has a non-trivial solution.*

   ii. *Suppose* $\mathcal{LS}(A,\, \mathbf{b})$ *is consistent. Further suppose the homogeneous system* $\mathcal{LS}(A,\, \mathbf{0}_m)$ *has a non-trivial solution. Then* $\mathcal{LS}(A,\, \mathbf{b})$ *has two distinct solutions.*

   iii. *Suppose* $\mathcal{LS}(A,\, \mathbf{b})$ *is consistent. Further suppose the homogeneous system* $\mathcal{LS}(A,\, \mathbf{0}_m)$ *has a non-trivial solution. Then* $\mathcal{LS}(A,\, \mathbf{b})$ *has infinitely many distinct solutions, in the sense of* ($\sharp$)*:*

   ($\sharp$) *There is an infinite sequence of column vectors* $\{\mathbf{t}_{(\nu)}\}_{\nu=0}^{\infty}$*, so that each* $\mathbf{t}_{(\nu)}$ *is a solution of* $\mathcal{LS}(A,\, \mathbf{b})$ *and the terms in the sequence* $\{\mathbf{t}_{(\nu)}\}_{\nu=0}^{\infty}$ *are pairwise distinct.*

   Remark. Note that in this course the word ‘number’ is understood as ‘real number’ or ‘complex number’.

   (b) Provide a counter-example against the statement below (and justify your answer):—

   Let A *be an* (3×4)*-matrix,* $\mathbf{b}$ *be a column vector with* 3 *entries.*

   Suppose the homogeneous system $\mathcal{LS}(A,\, \mathbf{0}_3)$ *has a non-trivial solution.*

   Then $\mathcal{LS}(A,\, \mathbf{b})$ *has two distinct solutions.*

6. *In this question,* $(S), (T), (U)$ *respectively stand for some linear systems, which, when written out explicitly, read as:—*

$$(S): \left\{ \begin{array}{ccccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & a_{14}x_4 & + & a_{15}x_5 & + & a_{16}x_6 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & a_{24}x_4 & + & a_{25}x_5 & + & a_{26}x_6 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & a_{34}x_4 & + & a_{35}x_5 & + & a_{36}x_6 & = & b_3 \\ a_{41}x_1 & + & a_{42}x_2 & + & a_{43}x_3 & + & a_{44}x_4 & + & a_{45}x_5 & + & a_{46}x_6 & = & b_4 \end{array} \right. ,$$

$$(T): \left\{ \begin{array}{ccccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & a_{14}x_4 & + & a_{15}x_5 & + & a_{16}x_6 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & a_{24}x_4 & + & a_{25}x_5 & + & a_{26}x_6 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & a_{34}x_4 & + & a_{35}x_5 & + & a_{36}x_6 & = & b_3 \end{array} \right. ,$$

$$(U): \left\{ \begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & a_{14}x_4 & + & a_{15}x_5 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & a_{24}x_4 & + & a_{25}x_5 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & a_{34}x_4 & + & a_{35}x_5 & = & b_3 \\ a_{41}x_1 & + & a_{42}x_2 & + & a_{43}x_3 & + & a_{44}x_4 & + & a_{45}x_5 & = & b_4 \end{array} \right. .$$

   The a_{ij}'s and the b_i's in the respective systems are the same numbers.

   Which of the statements below are true? Which of them false.

   Provide an appropriate justification for each answer (by giving a proof, or providing a counter-example, as appropriate).

   (a) *Suppose* (S) *is consistent. Then* (T) *is consistent.*

   (b) *Suppose* (T) *is consistent. Then* (S) *is consistent.*

   (c) *Suppose* (S) *is consistent. Then* (U) *is consistent.*

   (d) *Suppose* (U) *is consistent. Then* (S) *is consistent.*

7. For each statement below, determine whether it is true or false. If it is true, give a proof. If it is false, give a counter-example (and justify your answer).

   (a) *Let* A *be an* $(m \times n)$*-matrix,* C *be a* $(p \times m)$*-matrix, and* $\mathbf{d}$ *be a column vector with* m *entries.*

   Suppose $\mathcal{LS}(A,\, \mathbf{d})$ *is consistent.*

   Then $\mathcal{LS}(CA,\, C\mathbf{d})$ *is consistent.*

(b) *Let A, B be $(m \times n)$-matrices, and $\mathbf{c}, \mathbf{d}$ be column vectors with m entries.*
Suppose $\mathcal{LS}(A,\ \mathbf{c})$ and $\mathcal{LS}(B,\ \mathbf{d})$ are consistent.
Then $\mathcal{LS}(A+B,\ \mathbf{c}+\mathbf{d})$ is consistent.

(c) *Let A be an $(m \times n)$-matrix, B be a $(p \times n)$-matrix, $\mathbf{c}$ be a column vector with m entries, and $\mathbf{d}$ be a column vector with p entries.*
Suppose $\mathcal{LS}\left(\left[\begin{array}{c} A \\ \hline B \end{array}\right],\ \left[\begin{array}{c} \mathbf{c} \\ \hline \mathbf{d} \end{array}\right]\right)$ is consistent.
Then each of $\mathcal{LS}(A,\ \mathbf{c}), \mathcal{LS}(B,\ \mathbf{d})$ is consistent.

(d) *Let A be an $(m \times n)$-matrix, B be a $(p \times n)$-matrix, $\mathbf{c}$ be a column vector with m entries, and $\mathbf{d}$ be a column vector with p entries.*
Suppose each of $\mathcal{LS}(A,\ \mathbf{c}), \mathcal{LS}(B,\ \mathbf{d})$ is consistent.
Then $\mathcal{LS}\left(\left[\begin{array}{c} A \\ \hline B \end{array}\right],\ \left[\begin{array}{c} \mathbf{c} \\ \hline \mathbf{d} \end{array}\right]\right)$ is consistent.

(e) *Let A be an $(m \times n)$-matrix, B be an $(m \times p)$-matrix, and $\mathbf{c}, \mathbf{d}$ be column vectors with m entries.*
Suppose $\mathcal{LS}(A,\ \mathbf{c})$ and $\mathcal{LS}(B,\ \mathbf{d})$ are consistent.
Then $\mathcal{LS}([\ A \mid B\ ],\ \mathbf{c}+\mathbf{d})$ is consistent.

(f) *Let A be an $(m \times n)$-matrix, B be an $(p \times q)$-matrix, and $\mathbf{c}$ be a column vector with m entries, and $\mathbf{d}$ be a column vector with p entries.*
Suppose $\mathcal{LS}(A,\ \mathbf{c})$ and $\mathcal{LS}(B,\ \mathbf{d})$ are consistent.
Then $\mathcal{LS}\left(\left[\begin{array}{c|c} A & \mathcal{O}_{m\times q} \\ \hline \mathcal{O}_{p\times n} & B \end{array}\right],\ \left[\begin{array}{c} \mathbf{c} \\ \hline \mathbf{d} \end{array}\right]\right)$ is consistent.

8. Let A, B be $(n \times n)$-square matrices, and $\mathbf{c}$ is a column vector with n entries.
Suppose A, B are commuting, and $\mathcal{LS}(AB,\ \mathbf{c})$ is consistent.
Show that $\mathcal{LS}(A,\ \mathbf{c})$ and $\mathcal{LS}(B,\ c)$ are consistent.

9. Recall the notion of *idempotency*, whose definition is given below:—-

Let C be a square matrix.
We say that C is **idempotent** *if and only if $C^2 = C$.*

(a) Prove the statement (♯), with direct reference to the relevant definitions:—
(♯) *Let A be an idempotent $(n \times n)$-square matrix, and $\mathbf{c}$ be a column vector with n entries. Suppose the system $\mathcal{LS}(A,\ \mathbf{c} - A\mathbf{c})$ is consistent. Then $\mathcal{LS}(A,\ \mathbf{c})$ is consistent.*

(b) Is the converse of (♯) true? Justify your answer.
Remark. The converse of (♯) reads:—
Let A be an idempotent $(n \times n)$-square matrix, and $\mathbf{c}$ be a column vector with n entries. Suppose $\mathcal{LS}(A,\ \mathbf{c})$ is consistent. Then the system $\mathcal{LS}(A,\ \mathbf{c} - A\mathbf{c})$ is consistent.

10. Recall the notion of *involutoricy*, whose definition is given below:—-

Let C be a square matrix.
We say that C is **involutoric** *if and only if C^2 is the identity matrix.*

Prove the statement below, with direct reference to the relevant definitions:—
Let A be an $(n \times n)$-square matrix. Suppose A is involutory. Then, for any column vector b with n entries, the system $\mathcal{LS}(A,\ \mathbf{b})$ has one and only one solution.

11. (a) Prove the statement (♯):—
(♯) *Let A be an $(m \times n)$-matrix with real entries, and $\mathbf{b}$ be a column vector with m real entries.*
Suppose $\mathcal{LS}(A,\ \mathbf{b})$ is consistent and $\mathbf{b}$ is a solution of $\mathcal{LS}(A^t,\ \mathbf{0}_n)$.
Then $\mathbf{b} = \mathbf{0}_m$.
Remark. At some point of your argument you may need to use the following property of the real number system:
- *Suppose $c, \cdots, d$ are non-negative real numbers, and $c + \cdots + d = 0$. Then $c = \cdots = d = 0$.*

(b) Is the statement (♮) true or false? Justify your answer.
(♮) *Let A be a symmetric $(p \times p)$-square matrix with real entries, and $\mathbf{b}$ be a column vector with p real entries.*
Suppose $\mathbf{b}$ is a non-trivial solution of the homogeneous system $\mathcal{LS}(A,\ \mathbf{0}_p)$.
Then $\mathcal{LS}(A,\ \mathbf{b})$ is inconsistent.