2.1 Systems of linear equations.

0. *Assumed background.*

   - 1.1 *Matrices, matrix addition, and scalar multiplication for matrices.*
   - 1.2 *Matrix multiplication.*

   Abstract. We introduce:—

   - the notion of systems of linear equations,
   - matrix presentation and vector presentation of a system of linear equation,
   - augmented matrix representation of a system of linear equation,
   - solution of a system of linear equations,
   - the notions of consistency and inconsistency.

1. **Background.**
   Back in school days you encountered problems like:—

   (I) *Solve the simultaneous equations with unknowns x, y given by*

   $$\begin{cases} 2x + 3y = 5 \\ 3x - 4y = -1 \end{cases}$$

   Some of you might have also encountered more 'advanced' problems like:—

   (II) *Solve the simultaneous equations with unknowns x, y, z given by*

   $$\begin{cases} 2x + 3y + z = 6 \\ 3x - 4y + 2z = 1 \\ 4x + y - 6z = -1 \end{cases}$$

   Or perhaps:—

   (III) *Solve the simultaneous equations with unknowns x, y, z given by*

   $$\begin{cases} 2x + 3y - 2z = 3 \\ 3x - 5y + 4z = 2 \end{cases}$$

   Or perhaps:—

   (IV) *Determine whether the simultaneous equations with unknowns x, y given by*

   $$\begin{cases} 2x + 3y = 5 \\ 3x - 5y = -2 \\ 5x - 2y = 4 \end{cases}$$

   has any solution, and write down one solution if such such exists.

   These 'simultaneous equations with unknowns' are examples of systems of linear equations.

2. **Definition. (System of linear equations.)**

   (a) *A* **system of m linear equations with n unknowns** $x_1, x_2, x_3, \cdots, x_n$ *is a list of equations of the form*

   $$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases}$$

   in which the a_{ij}'s, b_k's are some fixed numbers, called the **givens** *in such a system.*

   (b) *From now on, denote such a system of linear equations by (S).*
   Let $t_1, t_2, \cdots, t_n$ be (fixed) numbers.
   We agree to say (and write)

'a **solution** *of the system* (S) *is given by* "$x_1 = t_1$ *and* $x_2 = t_2$ *and* $x_3 = t_3$ *and* $\cdots$ *and* $x_n = t_n$" '

if and only if upon the (simultaneous) substitution of

$$\text{`}x_1 = t_1 \text{ and } x_2 = t_2 \text{ and } x_3 = t_3 \text{ and } \cdots \text{ and } x_n = t_n\text{'}$$

into (S)*, the equalities (about numbers)*

$$\left\{\begin{array}{ccccccccccc} a_{11}t_1 & + & a_{12}t_2 & + & a_{13}t_3 & + & \cdots & + & a_{1n}t_n & = & b_1, \\ a_{21}t_1 & + & a_{22}t_2 & + & a_{23}t_3 & + & \cdots & + & a_{2n}t_n & = & b_2, \\ a_{31}t_1 & + & a_{32}t_2 & + & a_{33}t_3 & + & \cdots & + & a_{3n}t_n & = & b_3, \\ & & & & & & & & & \vdots & \\ a_{m1}t_1 & + & a_{m2}t_2 & + & a_{m3}t_3 & + & \cdots & + & a_{mn}t_n & = & b_m. \end{array}\right.$$

are yielded simultaneously.

(c) i. *We say* (S) *is* **consistent** *if and only if there is some solution for* (S)*.*

ii. *We say* (S) *is* **inconsistent** *if and only if there is no solution for* (S)*.*

3. **Observations in the context of the definition above.**

(a) Using the definition for matrix equality, we may re-express

$$(S): \left\{\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ & & & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

as an equation about vectors, in which the unknowns are $x_1, x_2, x_3, \cdots, x_n$:—

$$(S') \quad \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n \\ & & & & & & \vdots & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

(b) Next using the definition of matrix multiplication, we may further re-express (S) as:—

$$(S'') \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & & & \vdots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

which is an equation about matrices and column vectors, in which the unknowns in (S) are 're-organized' and 'consolidated' as the unknown column vector with entries $x_1, x_2, x_3, \cdots, x_n$.

(c) Sometimes we prefer exploiting the definition of vector addition and scalar multiplication to vectors so as to re-present (S') as:—

$$(S''') \quad x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix},$$

which is an equation about linear combinations of given column vectors in which the unknowns are the scalars to be combined with the given column vectors.

These observations allow for the re-formulation of the definition about system of linear equations, solutions, and consistency in terms of matrices and vectors.

4. **Definition. (System of linear equations, in terms of matrices and vectors.)**

(a) *A* **system of** m **linear equations with** n **unknowns** *is an equation of the form*

$$A\mathbf{x} = \mathbf{b},$$

in which A *is an* $(m \times n)$*-matrix, and* $\mathbf{b}$ *is a column vector with* m *entries, and in which* $\mathbf{x}$ *is a column vector with* n *entries, all being unknowns.*

i. *For such a system, we refer to* $A, \mathbf{b}$ *as its* **coefficient matrix** *and its* **vector of constants**.

ii.
- *When there is no need to mention* $\mathbf{x}$ *in the system '*$A\mathbf{x} = \mathbf{b}$*', we denote the system by* $\mathcal{LS}(A,\, \mathbf{b})$.
- *Where there is a need to mention* $\mathbf{x}$ *in the system '*$A\mathbf{x} = \mathbf{b}$*', we refer to it as the* **unknown vector** *for the system.*

iii. *The* $(m \times (n+1))$*-matrix* $[\ A \mid \mathbf{b}\ ]$ *is called the* **augmented matrix representation** *of the system* $\mathcal{LS}(A,\, \mathbf{b})$.

iv. *Suppose the columns of* A*, from left to right, are* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_n$ *respectively.*
We refer to the (column) vector equation with unknown scalars $x_1, x_2, x_3, \cdots, x_n$ *that reads*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

as the **vector presentation** *of the system* $\mathcal{LS}(A,\, \mathbf{b})$.

(b) *A (column) vector* $\mathbf{t}$ *with* n *entries is called a* **solution of** (*or* **for**) **the system** $\mathcal{LS}(A,\, \mathbf{b})$ *if and only if the equality of vectors '*$A\mathbf{t} = \mathbf{b}$*' holds.*

(c) i. *We say* $\mathcal{LS}(A,\, \mathbf{b})$ *is* **consistent** *if and only if there is some solution for the system* $\mathcal{LS}(A,\, \mathbf{b})$.

ii. *We say* $\mathcal{LS}(A,\, \mathbf{b})$ *is* **inconsistent** *if and only if there is no solution for the system* $\mathcal{LS}(A,\, \mathbf{b})$.

5. **Example (1).**

(a) Consider the system of linear equations

$$(S_1): \quad \left\{ \begin{array}{rcrcr} 2x_1 & + & 3x_2 & = & 5 \\ 3x_1 & - & 4x_2 & = & -1 \end{array} \right.$$

- (S_1) can be presented in terms of matrices and vectors as:—

$$\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

- The coefficient matrix and the vector of constants of (S_1) are respectively given by

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

- The augmented matrix representation of (S_1) is given by

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 3 & -4 & -1 \end{array} \right].$$

- The vector presentation of (S_1) is given by

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

- A solution of (S_1) is given by

$$\mathbf{t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

because we have the chain of equalities below:—

$$A\mathbf{t} = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & + & 3 \cdot 1 \\ 3 \cdot 1 & - & 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \mathbf{b}$$

It follows that (S_1) is consistent.

(b) Consider the system of linear equations

$$(S_2): \quad \left\{ \begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & x_3 & = & 6 \\ 3x_1 & - & 4x_2 & + & 2x_3 & = & 1 \\ 4x_1 & + & x_2 & - & 6x_3 & = & -1 \end{array} \right.$$

- (S_2) can be presented in terms of matrices and vectors as:—

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & -4 & 2 \\ 4 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}.$$

- The coefficient matrix and the vector of constants of (S_2) are respectively given by

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & -4 & 2 \\ 4 & 1 & -6 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}.$$

- The augmented matrix representation of (S_2) is given by

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 3 & -4 & 2 & 1 \\ 4 & 1 & -6 & -1 \end{array}\right].$$

- The vector presentation of (S_2) is given by

$$x_1 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}.$$

- A solution of (S_2) is given by

$$\mathbf{t} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

because we have the chain of equalities below:—

$$A\mathbf{t} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & -4 & 2 \\ 4 & 1 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & + & 3\cdot 1 & + & 1\cdot 1 \\ 3\cdot 1 & - & 4\cdot 1 & + & 2\cdot 1 \\ 4\cdot 1 & + & 1\cdot 1 & - & 6\cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} = \mathbf{b}$$

It follows that (S_2) is consistent.

(c) Consider the system of linear equations

$$(S_3): \quad \left\{ \begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & 8x_3 & = & -3 \\ x_1 & - & 2x_2 & + & 3x_3 & = & 2 \end{array} \right.$$

- (S_3) can be presented in terms of matrices and vectors as:—

$$\begin{bmatrix} 2 & 3 & -8 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

- The coefficient matrix and the vector of constants of (S_3) are respectively given by

$$A = \begin{bmatrix} 2 & 3 & -8 \\ 1 & -2 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

- The augmented matrix representation of (S_3) is given by

$$\left[\begin{array}{ccc|c} 2 & 3 & -8 & -3 \\ 1 & -2 & 3 & 2 \end{array}\right].$$

- The vector presentation of (S_3) is given by

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

- A solution of (S_3) is given by

$$\mathbf{t} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

because we have the chain of equalities below:—

$$A\mathbf{t} = \begin{bmatrix} 2 & 3 & -8 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & + & 3\cdot 1 & - & 8\cdot 1 \\ 1\cdot 1 & - & 2\cdot 1 & + & 3\cdot 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \mathbf{b}$$

It follows that (S_3) is consistent.

(d) Consider the system of linear equations

$$(S_4): \quad \left\{ \begin{array}{rcrcr} 2x_1 & + & 3x_2 & = & 5 \\ 3x_1 & - & 5x_2 & = & -2 \\ 5x_1 & - & 2x_2 & = & 4 \end{array} \right.$$

- (S_4) can be presented in terms of matrices and vectors as:—

$$\begin{bmatrix} 2 & 3 \\ 3 & -5 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}.$$

- The coefficient matrix and the vector of constants of (S_4) are respectively given by

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -5 \\ 5 & -2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}.$$

- The augmented matrix representation of (S_4) is given by

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 3 & -5 & -2 \\ 5 & -2 & 4 \end{array} \right].$$

- The vector presentation of (S_4) is given by

$$x_1 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -5 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}.$$

- We claim that (S_4) has no solution, and hence is inconsistent. Justification:—
  Suppose (S_4) had some solution, say, $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$.
  Then
  $$\begin{bmatrix} 2t_1 + 3t_2 \\ 3t_1 - 5t_2 \\ 5t_1 - 2t_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & -5 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = A\mathbf{t} = \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$
  Comparing the first and second entries, we have $2t_1 + 3t_2 = 5$ and $3t_1 - 5t_2 = -2$.
  Then $5t_1 - 2t_2 = 3$.
  Comparing the third entries, we have $5t_1 - 2t_2 = 4$.
  Then $3 = 5t_1 - 2t_2 = 4$, which is impossible.
  Hence, in the first place, (S_4) does not have any solution.

6. **Definition. (Homogeneous systems.)**

Let A be an $(m \times n)$-matrix.

(a) i. *The system $\mathcal{LS}(A, \mathbf{0}_m)$ is called the* **homogeneous system** *with coefficient matrix A.*

ii. *Whenever $\mathbf{c}$ is a non-zero column vector with m entries, the system $\mathcal{LS}(A, \mathbf{c})$ is called a* **non-homogeneous system***.*

(b) *A solution $\mathbf{t}$ of $\mathcal{LS}(A, \mathbf{0}_m)$ is called a* **non-trivial solution for the homogeneous system** *$\mathcal{LS}(A, \mathbf{0}_m)$ if and only if $\mathbf{t} \neq \mathbf{0}_n$.*

(c) *For any column vector $\mathbf{b}$ with m entries, we refer to $\mathcal{LS}(A, \mathbf{0}_m)$ as the* **homogeneous system associated to** *$\mathcal{LS}(A, \mathbf{b})$.*

Remark. The homogeneous system $\mathcal{LS}(A, \mathbf{0}_m)$ is guaranteed to have at least one solution, namely, the **trivial solution $\mathbf{0}_n$**. For this reason, whether $\mathcal{LS}(A, \mathbf{0}_m)$ possesses any non-trivial solution is interesting.

7. **Example (2).**

(a) Consider the system

$$(H_1): \quad \left\{ \begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & - & 2x_4 & = & 0 \\ 2x_1 & - & x_2 & - & 4x_3 & + & 3x_4 & = & 0 \\ 3x_1 & + & 2x_2 & - & 6x_3 & + & x_4 & = & 0 \end{array} \right.$$

- The system (H_1) is homogeneous, and its coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 2 & -1 & -4 & 3 \\ 3 & 2 & -6 & 1 \end{bmatrix}$$

- A non-trivial solution of (H_1) is given by

$$\mathbf{t} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

because we have the chain of equalities below:—

$$A\mathbf{t} = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 2 & -1 & -4 & 3 \\ 3 & 2 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 3\cdot 1 - 2\cdot 1 - 2\cdot 1 \\ 2\cdot 1 - 1\cdot 1 - 4\cdot 1 + 3\cdot 1 \\ 3\cdot 1 + 2\cdot 1 - 6\cdot 1 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Consider the system

$$(H_2): \begin{cases} 2x_2 - 3x_3 + 2x_4 - x_5 = 0 \\ x_1 - 4x_2 + 2x_3 - x_4 + 2x_5 = 0 \\ 3x_1 + x_2 - 2x_4 + 2x_5 = 0 \\ -2x_1 + 4x_3 + x_4 - 3x_5 = 0 \end{cases}$$

- The system (H_2) is homogeneous, and its coefficient matrix is given by

$$A = \begin{bmatrix} 0 & 2 & -3 & 2 & -1 \\ 1 & -4 & 2 & -1 & 2 \\ 3 & 1 & 0 & -2 & 2 \\ -2 & 0 & 4 & 1 & -3 \end{bmatrix}$$

- A non-trivial solution of (H_2) is given by

$$\mathbf{t} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

because we have the chain of equalities below:—

$$A\mathbf{t} = \begin{bmatrix} 0 & 2 & -3 & 2 & -1 \\ 1 & -4 & 2 & -1 & 2 \\ 3 & 1 & 0 & -2 & 2 \\ -2 & 0 & 4 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 + 2\cdot 1 - 3\cdot 1 + 2\cdot 1 - 1\cdot 1 \\ 1\cdot 1 - 4\cdot 1 + 2\cdot 1 - 1\cdot 1 + 2\cdot 1 \\ 3\cdot 1 + 1\cdot 1 + 0\cdot 1 - 2\cdot 1 + 2\cdot 1 \\ -2\cdot 1 \quad 0\cdot 1 + 4\cdot 1 + 1\cdot 1 - 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Consider the system

$$(H_3): \begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

- The system (H_3) is homogeneous, and its coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- We claim that (H_3) has no non-trivial solution. Justification:—

  Suppose $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$ is a solution of (H_3).

  [We want to show that $\mathbf{t}$ is the trivial solution of (H_3).]

  We have

$$\begin{bmatrix} t_1 + t_2 \\ t_2 + t_3 \\ t_1 + t_3 \\ t_1 + t_2 + t_3 \end{bmatrix} = A\mathbf{t} = \mathbf{0}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Comparing the first and fourth entries, we have $t_1 + t_2 = 0$ and $t_1 + t_2 + t_3 = 0$.
Then $t_3 = (t_1 + t_2 + t_3) - (t_1 + t_2) = 0$.
Comparing the second and fourth entries, we have $t_2 + t_3 = 0$ and $t_1 + t_2 + t_3 = 0$.
Then $t_1 = (t_1 + t_2 + t_3) - (t_2 + t_3) = 0$.
Comparing the third and fourth entries, we have $t_1 + t_3 = 0$ and $t_1 + t_2 + t_3 = 0$.
Then $t_2 = (t_1 + t_2 + t_3) - (t_1 + t_3) = 0$.
Therefore $\mathbf{t} = \mathbf{0}_3$.