1.8.1 Answers to Exercise.

1. (a) $J_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

   (b) $J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

   (c) $J_3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (d) $J = J_3 J_2 J_1 = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

   (e) $J' = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix}$.

2. (a) ρ_1 is $R_1 \longleftrightarrow R_2$.
   ρ_2 is $-2R_1 + R_2$.
   ρ_3 is $1R_1 + R_4$.
   ρ_4 is $R_2 \longleftrightarrow R_3$.
   ρ_5 is $-\frac{1}{5}R_3$.

   (b) A (4×4)-square matrix H for which the equality $A' = HA$ holds is given by the sequence of row operations $I_4 \xrightarrow{\rho_1}\xrightarrow{\rho_2}\xrightarrow{\rho_3}\xrightarrow{\rho_4}\xrightarrow{\rho_5} H$.

   It is given by $H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/5 & 2/5 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

3. (a) ρ_1 is $R_1 \leftrightarrow R_2$.
   ρ_2 is $-4R_1 + R_5$.
   ρ_3 is $-\frac{1}{3}R_3$.

   (b) $B_3 \xrightarrow{1R_2+R_1} B_2 \xrightarrow{-2R_3+R_4} B_1$

   (c) $K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

   (d)

$$I_5 \xrightarrow{2R_3+R_4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_2+R_1} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_3+R_1} \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_2} \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{1R_4+R_2} \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = H$$

4. (a) $\begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} -8 & 5 & 2 \\ 7 & -4 & -2 \\ -3 & 2 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -3 & 5 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix}$

5. (a) $J_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $J_4 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (b) $B_4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$.

   (c) $B_1A_1 \xrightarrow{-1R_1+R_2} B_2A_1 \xrightarrow{-2R_1+R_3} B_3A_1 \xrightarrow{-1R_2+R_3} B_4A_1 \xrightarrow{-1R_2+R_1} A_1 \xrightarrow{-1R_1+R_2} A_2 \xrightarrow{-2R_1+R_3} A_3 \xrightarrow{-1R_2+R_3} A_4 \xrightarrow{-1R_2+R_1} A_5$

6. (a) $G = \begin{bmatrix} 1+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3 & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & 1 & 0 & 0 \\ \alpha_2 & 0 & 1 & 0 \\ \alpha_3 & 0 & 0 & 1 \end{bmatrix}$.

   (b) $H = \begin{bmatrix} 1 & -\beta_1 & -\beta_2 & -\beta_3 \\ -\alpha_1 & 1+\alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 \\ -\alpha_2 & \alpha_2\beta_1 & 1+\alpha_2\beta_2 & \alpha_3\beta_3 \\ -\alpha_3 & \alpha_3\beta_1 & \alpha_3\beta_2 & 1+\alpha_3\beta_3 \end{bmatrix}$

7. *Comment.*

   Contrast with the argument for proving the statement with row-operation matrices and that for proving the statement with row operations alone.

8. *Outline of argument.*

   By assumption, there are some row-operation matrices, all of them $(m \times m)$-square matrices, say, $G_1, G_2, \cdots, G_k$ such that $A' = G_k \cdots G_2G_1A$.

   Also by assumption, there are some row-operation matrices, all of them $(p \times p)$-square matrices, say, $H_1, H_2, \cdots, H_\ell$ such that $B' = H_\ell \cdots H_2H_1B$.

   Define the $((m+p) \times (m+p))$-matrices $J_1, J_2, \cdots, J_k, J_{k+1}, J_{k+2}, \cdots, J_{k+\ell}$ by

$$\begin{cases} J_s = \left[\begin{array}{c|c} G_s & \mathcal{O}_{m\times p} \\ \hline \mathcal{O}_{p\times m} & I_p \end{array}\right] & \text{for each } s = 1, 2, \cdots, k \\ J_{k+t} = \left[\begin{array}{c|c} I_m & \mathcal{O}_{m\times p} \\ \hline \mathcal{O}_{p\times m} & H_t \end{array}\right] & \text{for each } t = 1, 2, \cdots, \ell \end{cases}$$

   It follows that $\left[\begin{array}{c} A' \\ \hline B' \end{array}\right] = J_{k+\ell} \cdots J_{k+2}J_{k+1}J_k \cdots J_2J_1 \left[\begin{array}{c} A \\ \hline B \end{array}\right]$.

   Note that each of $J_1, J_2, \cdots, J_{k+\ell}$ is a row-operation matrix by construction.

9. (a) ——

   (b) An appropriate choice is to take $A' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A'' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B'' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (Note that $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.)