1.8.1 Exercise: Row operations and matrix multiplication.

1. Let A_1, A_2, A_3, A_4 be (3×5)-matrices and suppose they are row-equivalent under the sequence of row operations below:
$$A_1 \xrightarrow{2R_1+R_3} A_2 \xrightarrow{R_2 \leftrightarrow R_3} A_3 \xrightarrow{3R_1} A_4.$$
   (a) Write down a (3×3)-matrix J_1 for which $A_2 = J_1 A_1$.
   (b) Write down a (3×3)-matrix J_2 for which $A_3 = J_2 A_2$.
   (c) Write down a (3×3)-matrix J_3 for which $A_4 = J_3 A_3$.
   (d) Write down a (3×3)-matrix J for which the equality $A_4 = J A_1$ holds.
   (e) Write down a (3×3)-matrix J' for which the equality $A_1 = J' A_4$ holds.

2. A sequence of row operations $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ joining two (4×6)-matrices A, A' is given below:—
$$A = \begin{bmatrix} 2 & 1 & -1 & 4 & 3 & 8 \\ 1 & 3 & 0 & 8 & -4 & -4 \\ 0 & 1 & 1 & 4 & -3 & -4 \\ -1 & -2 & 3 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{\rho_1} \begin{bmatrix} 1 & 3 & 0 & 8 & -4 & -4 \\ 2 & 1 & -1 & 4 & 3 & 8 \\ 0 & 1 & 1 & 4 & -3 & -4 \\ -1 & -2 & 3 & 0 & 0 & -1 \end{bmatrix}$$
$$\xrightarrow{\rho_2} \begin{bmatrix} 1 & 3 & 0 & 8 & -4 & -4 \\ 0 & -5 & -1 & -12 & 11 & 16 \\ 0 & 1 & 1 & 4 & -3 & -4 \\ -1 & -2 & 3 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{\rho_3} \begin{bmatrix} 1 & 3 & 0 & 8 & -4 & -4 \\ 0 & -5 & -1 & -12 & 11 & 16 \\ 0 & 1 & 1 & 4 & -3 & -4 \\ 0 & 1 & 3 & 8 & -4 & -5 \end{bmatrix}$$
$$\xrightarrow{\rho_4} \begin{bmatrix} 1 & 3 & 0 & 8 & -4 & -4 \\ 0 & 1 & 1 & 4 & -3 & -4 \\ 0 & -5 & -1 & -12 & 11 & 16 \\ 0 & 1 & 3 & 8 & -4 & -5 \end{bmatrix} \xrightarrow{\rho_5} \begin{bmatrix} 1 & 3 & 0 & 8 & -4 & -4 \\ 0 & 1 & 1 & 4 & -3 & -4 \\ 0 & 1 & 1/5 & 12/5 & -11/5 & -16/5 \\ 0 & 1 & 3 & 8 & -4 & -5 \end{bmatrix} = A'$$
   (a) Provide appropriate labels for the row operations $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$.
   (b) Write down an appropriate (4×4)-square matrix H for which the equality $A' = HA$ holds.

3. A sequence of row operations joining the (5×7)-matrices $A, B_1, B_2, B_3, B_4, B_5, B_6$ is given below. The symbols ρ_1, ρ_2, ρ_3 stands for some row operations in this sequence.
$$A = \begin{bmatrix} 3 & 2 & 1 & -2 & 1 & -2 & -3 \\ 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 1 & 2 & 3 & -1 & 2 & -2 & 1 \\ -1 & -1 & -1 & -2 & -9 & 2 & -2 \\ 4 & 3 & 2 & -2 & 4 & -3 & -2 \end{bmatrix} \xrightarrow{\rho_1} \begin{bmatrix} 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 3 & 2 & 1 & -2 & 1 & -2 & -3 \\ 1 & 2 & 3 & -1 & 2 & -2 & 1 \\ -1 & -1 & -1 & -2 & -9 & 2 & -2 \\ 4 & 3 & 2 & -2 & 4 & -3 & -2 \end{bmatrix}$$
$$\longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 0 & -1 & -2 & -2 & -8 & 1 & -6 \\ 0 & 1 & 2 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -2 & -6 & 1 & -1 \\ 4 & 3 & 2 & -2 & 4 & -3 & -2 \end{bmatrix} \xrightarrow{\rho_2} \begin{bmatrix} 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 0 & -1 & -2 & -2 & -8 & 1 & -6 \\ 0 & 1 & 2 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -2 & -6 & 1 & -1 \\ 0 & -1 & -2 & -2 & -8 & 1 & -6 \end{bmatrix}$$
$$\longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 0 & 1 & 2 & 2 & 8 & -1 & 6 \\ 0 & 0 & 0 & -3 & -9 & 0 & -6 \\ 0 & 0 & 0 & -2 & -6 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\rho_3} B_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 0 & 1 & 2 & 2 & 8 & -1 & 6 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & -2 & -6 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{2R_3+R_4} B_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 0 & 1 & 2 & 2 & 8 & -1 & 6 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_2+R_1} B_3 = \begin{bmatrix} 1 & 0 & -1 & -2 & -5 & 0 & -5 \\ 0 & 1 & 2 & 2 & 8 & -1 & 6 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{2R_3+R_1} B_4 = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 2 & 8 & -1 & 6 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_3+R_2} B_5 = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{1R_4+R_2} B_6 = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
   (a) Provide appropriate labels for the row operations ρ_1, ρ_2, ρ_3.

(b) Fill in the blanks (I), (III) with row operations and the blank (II) with a matrix so as to give a sequence of row operations which starts from B_3 and ends at B_1:

$$B_3 \quad \underset{\text{(I)}}{\underline{\qquad\qquad}} \quad \underset{\text{(II)}}{\underline{\qquad}} \quad \underset{\text{(III)}}{\underline{\qquad\qquad}} \quad B_1$$

(c) Write down an appropriate (5×5)-square matrix K for which the equality $B_5 = KB_4$ holds.

(d) Is there a sequence of row operations which joins I_5 and some (5×5)-square matrix H for which the equality $B_6 = HB_1$ holds?

If *yes*, write down one such sequence, giving all row operations and matrices (including H of course) in the sequence explicitly.

If *no*, simply write '*there is no such sequence*'.

4. For the matrices labelled A, B in each part below, find an appropriate square matrix H for which $B = HA$.

(a) Here A, B are (3×4)-matrices, and are row-equivalent under the sequence of row operations:

$$A \xrightarrow{-1R_1+R_2} \xrightarrow{-2R_1+R_3} \xrightarrow{-2R_2+R_3} \xrightarrow{-1R_3} \xrightarrow{-2R_2+R_1} \xrightarrow{-1R_3+R_2} B$$

(b) Here A, B are (3×4)-matrices, and are row-equivalent under the sequence of row operations:

$$A \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{-1R_1} \xrightarrow{-2R_1+R_3} \xrightarrow{-3R_2+R_3} \xrightarrow{-2R_2+R_1} B$$

(c) Here A, B are (3×6)-matrices, and are row-equivalent under the sequence of row operations:

$$A \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{2R_1+R_3} \xrightarrow{-3R_2+R_3} \xrightarrow{-2R_2+R_1} \xrightarrow{2R_3+R_1} \xrightarrow{-2R_3+R_2} B$$

(d) Here A, B are (4×5)-matrices, and are row-equivalent under the sequence of row operations:

$$A \xrightarrow{-1R_1+R_2} \xrightarrow{-3R_1+R_3} \xrightarrow{-1R_1+R_4} \xrightarrow{-1R_2} \xrightarrow{4R_2+R_3} \xrightarrow{3R_2+R_4} \xrightarrow{R_3 \leftrightarrow R_4} \xrightarrow{4R_3+R_4} \xrightarrow{-2R_2+R_1} \xrightarrow{-1R_3+R_1} B$$

5. Let A_1, A_2, A_3, A_4, A_5 be (3×4)-matrices, and suppose they are row-equivalent under the sequence of row operations below:

$$A_1 \xrightarrow{-1R_1+R_2} A_2 \xrightarrow{-2R_1+R_3} A_3 \xrightarrow{-1R_2+R_3} A_4 \xrightarrow{-1R_2+R_1} A_5$$

(a) Write down appropriate (3×3)-square matrices J_1, J_2, J_3, J_4 for which the equalities $A_2 = J_1A_1$, $A_3 = J_2A_2$, $A_4 = J_3A_3$, $A_5 = J_4A_4$.

(b) Let B_1, B_2, B_3, B_4 be (3×3)-square matrix, and suppose $[\ A_1 \mid B_1\ ]$, $[\ A_2 \mid B_2\ ]$, $[\ A_3 \mid B_3\ ]$, $[\ A_4 \mid B_4\ ]$, $[\ A_5 \mid I_3\ ]$ are row-equivalent under the sequence of row operations below:

$$[\ A_1 \mid B_1\ ] \xrightarrow{-1R_1+R_2} [\ A_2 \mid B_2\ ] \xrightarrow{-2R_1+R_3} [\ A_3 \mid B_3\ ] \xrightarrow{-1R_2+R_3} [\ A_4 \mid B_4\ ] \xrightarrow{-1R_2+R_1} [\ A_5 \mid I_3\ ]$$

Find B_1, B_2, B_3, B_4 explicitly.

(c) Is it true that B_1A_1 is row-equivalent to A_5? If yes, provide an appropriate sequence of row operations joining B_1A_1 and A_5, giving the row operations involved in the sequence explicitly. If no, explain why B_1A_1 is not row-equivalent to A_5.

6. Let A, B be matrices with four rows. Let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ be numbers.

Suppose A, B are row-equivalent, and are joint by the sequence of row operations

$$A \xrightarrow{\alpha_1R_1+R_2} \xrightarrow{\alpha_2R_1+R_3} \xrightarrow{\alpha_3R_1+R_4} \xrightarrow{\beta_1R_2+R_1} \xrightarrow{\beta_2R_3+R_1} \xrightarrow{\beta_3R_4+R_1} B.$$

(a) Write down a (4×4)-matrix G (in terms of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$) for which the equality $B = GA$ holds.

(b) Write down a (4×4)-matrix H (in terms of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$) for which the equality $A = HB$ holds.

7. Prove the statements below:—

Let A, A' be $(m \times n)$-matrices. Suppose A is row-equivalent to A'.

Then AB is row-equivalent to $A'B$ for any $(n \times p)$-matrix B.

8. Prove the statements below:—

Let A, A' be $(m \times n)$-matrices, and B, B' be $(p \times n)$-matrices. Suppose A is row-equivalent to A', and B is row-equivalent to B'.

Then $\left[\begin{array}{c} A \\ \hline B \end{array}\right]$ *is row-equivalent to* $\left[\begin{array}{c} A' \\ \hline B' \end{array}\right]$.

9. (a) Prove the statements below:—

   Let A, B be $(m \times n)$-matrices.

   Suppose $A = [\ A' \mid A''\ ]$, and $B = [\ B' \mid B''\ ]$, in which:—

   - *A' are the first p columns of A, and A'' are the last $n - p$ columns of A, and*
   - *B' are the first p columns of B, and B'' are the last $n - p$ columns of B.*

   Then A is row-equivalent to B if and only if A', A'' are row-equivalent to B', B'' under some common sequence of row operations.

   (b) Give a counter-example against the statement below (and justify your answer).

   Let A, B be $(m \times n)$-matrices.

   Suppose $A = [\ A' \mid A''\ ]$, and $B = [\ B' \mid B''\ ]$, in which:—

   - *A' are the first p columns of A, and A'' are the last $n - p$ columns of A, and*
   - *B' are the first p columns of B, and B'' are the last $n - p$ columns of B.*

   Suppose A' is row-equivalent to B', and A'' is row-equivalent B''. Then A is row-equivalent to B.