

## 1.8 Row operations and matrix multiplication.

0. *Assumed background.*

- 1.2 Matrix multiplication.
- 1.7 Row operations on matrices.

*Abstract.* We introduce:—

- the notion of row operations matrices,
- the ‘equivalence’ of application of row operations and left-multiplication by row operation matrices.

### 1. Definition. (‘Standard base’ for a ‘vector space of matrices’)

Fix any positive integers  $p, q$ .

For each  $i = 1, \dots, p, j = 1, \dots, q$ , we define the  $(p \times q)$ -matrix  $E_{i,j}^{p,q}$  to be the  $(p \times q)$ -matrix whose  $(i, j)$ -th entry is 1 and whose other entries are all 0.

**Remark.** According to definition, there are altogether  $pq$  matrices  $E_{i,j}^{p,q}$  as  $i, j$  vary. They are collectively referred to as the ‘standard base’ for the ‘vector space of  $(p \times q)$ -matrices’.

### 2. Example (1). (‘Standard base’ for various ‘vector spaces of matrices’)

$$\begin{aligned} \text{(a)} \quad E_{1,1}^{2,3} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,2}^{2,3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,3}^{2,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \\ E_{2,1}^{2,3} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_{2,2}^{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, E_{2,3}^{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \\ \text{(b)} \quad E_{1,1}^{3,3} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,2}^{3,3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,3}^{3,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ E_{2,1}^{3,3} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{2,2}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{2,3}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \\ E_{3,1}^{3,3} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_{3,2}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, E_{3,3}^{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

### 3. Lemma (1).

Let  $p, q$  be positive integers. Suppose  $s, t$  are integers between 1 and  $p$ .

Let  $A$  be a  $(p \times q)$ -matrix, whose  $(i, j)$ -th entry is denoted by  $a_{ij}$ .

Then  $E_{s,t}^{p,p}A$  is the  $(p \times q)$ -matrix whose  $s$ -th row is  $[a_{t1} \ a_{t2} \ \dots \ a_{tq}]$ , and whose every other entry is 0.

**Remark.** In plain words, multiplying  $E_{s,t}^{p,p}$  to  $A$  from the left results in a new  $(p \times q)$ -matrix in which:—

- the  $s$ -th row is formed by the  $t$ -th row of  $A$ , and
- every other row is ‘set’ to zero.

### 4. Example (2). (Illustrations of Lemma (1).)

(a) Suppose  $A$  is the  $(3 \times 4)$ -matrix whose  $(i, j)$ -th entry is given by  $a_{ij}$ . Then:

$$\begin{aligned} \text{i.} \quad E_{1,2}^{3,3}A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \\ \text{ii.} \quad E_{3,1}^{3,3}A &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}. \\ \text{iii.} \quad E_{3,3}^{3,3}A &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \end{aligned}$$

(b) Suppose  $A$  is the  $(4 \times 6)$ -matrix whose  $(i, j)$ -th entry is given by  $a_{ij}$ . Then:

$$\text{i.} \quad E_{2,4}^{4,4}A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{ii. } E_{3,2}^{4,4}A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{iii. } E_{4,1}^{4,4}A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}.$$

## 5. Proof of Lemma (1).

For convenience, denote the  $(g, h)$ -th entry of  $E_{s,t}^{p,p}$  by  $\varepsilon_{gh}$ .

$$\text{So } \varepsilon_{gh} = \begin{cases} 1 & \text{if } g = s \text{ and } h = t \\ 0 & \text{if } g \neq s \text{ or } h \neq t \end{cases}$$

By definition, the  $(g, k)$ -th entry of  $E_{s,t}^{p,p}A$  is given by

$$\varepsilon_{g1}a_{1k} + \varepsilon_{g2}a_{2k} + \cdots + \varepsilon_{st}a_{tk} + \cdots + \varepsilon_{gp}a_{pk}.$$

- [We first focus on the  $s$ -th row of  $E_{s,t}^{p,p}A$ .]

For each  $k = 1, 2, \dots, q$ , the  $(s, k)$ -th entry of  $E_{s,t}^{p,p}A$  is the product of the  $s$ -th row of  $E_{s,t}^{p,p}$  and the  $k$ -th column of  $A$ , and therefore is given by

$$\underbrace{\varepsilon_{s1}a_{1k} + \varepsilon_{s2}a_{2k} + \cdots + \varepsilon_{st}a_{tk} + \cdots + \varepsilon_{sp}a_{pk}}_{\text{all terms being 0 except possibly the term involving } \varepsilon_{st}} = \varepsilon_{st}a_{tk} = a_{tk}.$$

Hence the  $s$ -th row of  $E_{s,t}^{p,p}A$  is  $[a_{t1} \ a_{t2} \ \cdots \ a_{tq}]$ .

- [We now turn to every other row of  $E_{s,t}^{p,p}A$ .]

Whenever  $g \neq s$ , we have  $\varepsilon_{gh} = 0$  for each  $h$ . Then, no matter which  $k$  is, the  $(g, k)$ -th entry of  $E_{s,t}^{p,p}A$  is a sum of  $p$  copies of 0's, and hence is 0.

## 6. Lemma (2).

Suppose  $A$  is a matrix with  $p$  rows, and  $i, k$  are integers between 1 and  $p$ . Then:—

- For any number  $\alpha$ , the application of the row operation  $\alpha R_i + R_k$  on  $A$  results in  $(I_p + \alpha E_{k,i}^{p,p})A$ .
- For any non-zero number  $\beta$ , the application of the row operation  $\beta R_k$  on  $A$  results in  $(I_p + (\beta - 1)E_{k,k}^{p,p})A$ .
- The application of the row operation  $R_i \leftrightarrow R_k$  on  $A$  results in  $(I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p})A$ .

**Proof of Lemma (2).** Exercise. (Straightforward calculation with the help of Lemma (1).)

## 7. Example (3). (Illustrations of Lemma (2).)

Suppose  $A$  is the  $(3 \times 4)$ -matrix whose  $(i, j)$ -th entry is given by  $a_{ij}$ . Then we have these applications of row operations:—

- 

$$A \xrightarrow{4R_2+R_1} \begin{bmatrix} 4a_{21} + a_{11} & 4a_{22} + a_{12} & 4a_{23} + a_{13} & 4a_{24} + a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\ = 4 \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = 4E_{1,2}^{3,3}A + A = (I_3 + 4E_{1,2}^{3,3})A$$

- 

$$A \xrightarrow{5R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 5a_{21} & 5a_{22} & 5a_{23} & 5a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\ = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} = A + 4E_{2,2}^{3,3}A = (I_3 + 4E_{2,2}^{3,3})A$$

(c)

$$\begin{aligned}
A \xrightarrow{R_1 \leftrightarrow R_3} & \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \\
= & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\
& + \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \\
= & A - E_{1,1}^{3,3}A - E_{3,3}^{3,3}A + E_{1,3}^{3,3}A + E_{3,1}^{3,3}A = (I_3 - E_{1,1}^{3,3} - E_{3,3}^{3,3} + E_{1,3}^{3,3} + E_{3,1}^{3,3})A.
\end{aligned}$$

## 8. Symbols and labels associated with Lemma (2).

In symbolic terms, what we have described in Lemma (2) is the validity, for every matrix with  $p$  rows, say,  $A$ , of the applications of the row operations below:—

- $A \xrightarrow{\alpha R_i + R_k} (I_p + \alpha E_{k,i}^{p,p})A$ .
- $A \xrightarrow{\beta R_k} (I_p + (\beta - 1)E_{k,k}^{p,p})A$ .
- $A \xrightarrow{R_i \leftrightarrow R_k} (I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p})A$ .

Because of their behaviour, it makes sense to give special labels for these matrices. From now on:—

- We label  $I_p + \alpha E_{k,i}^{p,p}$  as  $M[\alpha R_i + R_k]$ .
- We label  $I_p + (\beta - 1)E_{k,k}^{p,p}$  as  $M[\beta R_k]$ .
- We label  $I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p}$  as  $M[R_i \leftrightarrow R_k]$ .

Despite their seemingly complex formulae, these matrices are easy to write down explicitly, because they are the resultants of applications of the respective row operations on the identity matrix:—

- $I_p \xrightarrow{\alpha R_i + R_k} I_p + \alpha E_{k,i}^{p,p} = M[\alpha R_i + R_k]$ .
- $I_p \xrightarrow{\beta R_k} I_p + (\beta - 1)E_{k,k}^{p,p} = M[\beta R_k]$ .
- $I_p \xrightarrow{R_i \leftrightarrow R_k} I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p} = M[R_i \leftrightarrow R_k]$ .

## 9. Definition. (Row-operation matrices.)

- Let  $\rho$  be a row operation on matrices with  $p$  rows. The matrix  $M[\rho]$  is called the **row-operation matrix** (or **elementary matrix**) associated with  $\rho$ .
- A **row-operation matrix** (or **elementary matrix**) of size  $p$  is a  $(p \times p)$ -square matrix which is the row-operation matrix associated with some row-operation  $\rho$  on matrices with  $p$  rows.

## 10. Example (4). (Illustrations on row-operation matrices.)

- For the row operation  $3R_4 + R_2$  on matrices with 5 rows, its row-operation matrix  $M[3R_4 + R_2]$  is given by

$$I_5 \xrightarrow{3R_4 + R_2} M[3R_4 + R_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- For the row operation  $4R_3$  on matrices with 5 rows, its row-operation matrix  $M[4R_3]$  is given by

$$I_5 \xrightarrow{4R_3} M[4R_3] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) For the row operation  $R_2 \leftrightarrow R_5$  on matrices with 5 rows, its row-operation matrix  $M[R_2 \leftrightarrow R_5]$  is given by

$$I_5 \xrightarrow{R_2 \leftrightarrow R_5} M[R_2 \leftrightarrow R_5] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

11. **Theorem (3).** (**‘Dictionary’ between row operations and matrix multiplication from the left.**)

Let  $p, q$  be positive integers.

For any row operation  $\rho$  on matrices with  $p$  rows, there exists some unique  $(p \times p)$ -square matrix  $H$ , namely,  $H = M[\rho]$ , such that for any matrix with  $p$  rows, say,  $A$ , the matrix  $HA$  is the resultant of the application of  $\rho$  on the matrix  $A$ .

**Remark.** Theorem (3) below describes a ‘dictionary’ between the collection of all row operations on matrices with  $p$  rows and the collection of all row-operation matrices of size  $p$ .

This ‘dictionary’ tells us the ‘application of row operations’ and the ‘multiplication from the left by row-operation matrices’ are two ways of thinking about the same thing.

To write

$$\text{‘the row operation } A \xrightarrow{\rho} A' \text{ is valid’}$$

is the same as writing

$$\text{‘the equality } A' = M[\rho]A \text{ holds’}.$$

12. **Proof of Theorem (3).**

The ‘existence’ part of the argument follows immediately from Lemma (2). We only need to handle the ‘uniqueness’ part of the argument.

Suppose  $\rho$  is a row operation on  $(p \times q)$  matrix, and  $H$  is a  $(p \times p)$ -square matrix for which the following holds:—

- For any  $(p \times q)$ -matrix  $A$ , the application of  $\rho$  on  $A$  results in  $HA$ .

Now, in particular, the application of  $\rho$  on  $I_p$  results in  $H$ .

Recall that by the definition of  $M[\rho]$ , the application of  $\rho$  on  $I_p$  results in  $M[\rho]$ .

It follows that  $H = M[\rho]$ .

13. The ‘dictionary’ informs us that to apply a sequence of row operations on a matrix is the same as to multiply the product of the corresponding row-operation matrices (put in an appropriate order) to the same matrix from the left.

**Theorem (4).** (**Corollary to Theorem (3).**)

Let  $A_1, A_2, \dots, A_N$  be  $(p \times q)$ -matrices.

Suppose  $A_1, A_2, \dots, A_N$  are row-equivalent to each other, and  $A_1$  is joint to  $A_N$  by some sequence of row operations  $\rho_1, \rho_2, \dots, \rho_{N-1}$ :

$$A_1 \xrightarrow{\rho_1} A_2 \xrightarrow{\rho_2} A_3 \xrightarrow{\rho_3} \dots \xrightarrow{\rho_{N-2}} A_{N-1} \xrightarrow{\rho_{N-1}} A_N$$

Then the equalities

$$\begin{aligned} A_2 &= M[\rho_1]A_1, & A_3 &= M[\rho_2]A_2, & \dots &, & A_{N-1} &= M[\rho_{N-2}]A_{N-2}, & A_N &= M[\rho_{N-1}]A_{N-1}, \\ A_N &= M[\rho_{N-1}]M[\rho_{N-2}] \cdot \dots \cdot M[\rho_3]M[\rho_2]M[\rho_1]A_1 \end{aligned}$$

hold.

**Proof of Theorem (4).** This is an immediate consequence of Theorem (3).

14. **Comment on the content of Theorem (4).**

Implicit in this result is the information on:—

- how we can obtain the product

$$M[\rho_{N-1}]M[\rho_{N-2}] \cdot \dots \cdot M[\rho_3]M[\rho_2]M[\rho_1]$$

without having to perform matrix multiplication.

This product is the resultant of the application of the sequence of row operations  $\rho_1, \rho_2, \rho_3, \dots, \rho_{N-2}, \rho_{N-1}$  on the identity matrix  $I_p$ :

$$I_p \xrightarrow{\rho_1} M[\rho_1] \xrightarrow{\rho_2} M[\rho_2]M[\rho_1] \xrightarrow{\rho_3} M[\rho_3]M[\rho_2]M[\rho_1] \\ \xrightarrow{\rho_4} \dots \xrightarrow{\rho_{N-2}} M[\rho_{N-2}] \cdots M[\rho_3]M[\rho_2]M[\rho_1] \xrightarrow{\rho_{N-1}} M[\rho_{N-1}]M[\rho_{N-2}] \cdots M[\rho_3]M[\rho_2]M[\rho_1]$$

15. **Example (5). (Illustrations on Theorem (4).)**

(a) The sequence of row operations below joins  $A$  and  $A''$ :

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2+R_1} A'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Then  $A'' = H_2H_1A$ , in which

$$H_1 = M[1R_1 + R_2] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = M[2R_2 + R_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So  $A'' = HA$ , in which  $H = H_2H_1 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

(b) The sequence of row operations below joins  $B$  and  $B''$ :

$$B = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} B' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} B'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Then  $B'' = H_2H_1B$ , in which

$$H_1 = M[4R_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = M[-2R_1] = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So  $B'' = HB$ , in which  $H = H_2H_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

(c) The sequence of row operations below joins  $C$  and  $C''$ :

$$C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}.$$

Then  $C'' = H_2H_1C$ , in which  $H_1 = M[R_1 \leftrightarrow R_2] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $H_2 = M[R_2 \leftrightarrow R_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ .

So  $C'' = HC$ , in which  $H = H_2H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ .

(d) The sequence of row operations below joins  $A_1$  and  $A_4$ :

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} A_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_3} A_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} A_4 = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Then  $A_4 = H_3H_2H_1A_1$ , in which

$$H_1 = M[1R_1 + R_2] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H_2 = M[2R_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad H_3 = M[R_1 \leftrightarrow R_3] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

So  $A_4 = HA_1$ , in which  $H = H_3H_2H_1 = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ .

16. **Theorem (5).** (Matrix of ‘reverse row-operation’.)

Suppose  $\rho$  is a row operation on matrices with  $p$  rows, and  $\tilde{\rho}$  is its corresponding ‘reverse row operation’ on matrices with  $p$  rows.

Then  $M[\tilde{\rho}]M[\rho] = I_p$  and  $M[\rho]M[\tilde{\rho}] = I_p$ .

**Proof of Theorem (5).**

Suppose  $\rho$  is a row operation on matrices with  $p$  rows, and  $\tilde{\rho}$  is its corresponding ‘reverse row operation’ on matrices with  $p$  rows.

By the definition of row-operation matrices, we have the sequence of row operations:

$$I_p \xrightarrow{\rho} M[\rho] \xrightarrow{\tilde{\rho}} M[\tilde{\rho}]M[\rho].$$

As  $\tilde{\rho}$  is the ‘reverse row operation’ for  $\rho$ ,  $M[\tilde{\rho}]M[\rho] = I_p$ .

Note that  $\rho$  is the ‘reverse row operation’ for  $\tilde{\rho}$ . Repeating the argument, with the roles of  $\rho, \tilde{\rho}$  interchanged, we deduce that  $M[\rho]M[\tilde{\rho}] = I_p$ .

**Remark.** The ‘formulae’ for the row-operation matrices of ‘reverse row operations’ for row operations of various types are explicitly described below:—

Row operation $\rho$ on matrices with $p$ rows.	Row-operation matrix $M[\rho]$ .	‘Reverse row operation’ $\tilde{\rho}$ for $\rho$ .	Row-operation matrix $M[\tilde{\rho}]$ .
$\alpha R_i + R_k.$	$I_p + \alpha E_{k,i}^{p,p}.$	$-\alpha R_i + R_k.$	$I_p - \alpha E_{k,i}^{p,p}.$
$\beta R_k.$	$I_p + (\beta - 1)E_{k,k}^{p,p}.$	$(1/\beta)R_k.$	$I_p + (1/\beta - 1)E_{k,k}^{p,p}.$
$R_i \leftrightarrow R_k.$	$I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p}.$	$R_i \leftrightarrow R_k.$	$I_p - E_{i,i}^{p,p} - E_{k,k}^{p,p} + E_{i,k}^{p,p} + E_{k,i}^{p,p}.$

17. **Example (6).** (Row operation matrices and matrices of corresponding ‘reverse row operations’.)

(a) For the row operation  $3R_4 + R_2$  on matrices with 5 rows, its row operation matrix  $M[3R_4 + R_2]$  is given by

$$I_5 \xrightarrow{3R_4+R_2} M[3R_4 + R_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding ‘reverse row operation’ on matrices with 5 rows is  $-3R_4 + R_2$ , and its row operation matrix  $M[-3R_4 + R_2]$  is given by

$$I_5 \xrightarrow{-3R_4+R_2} M[-3R_4 + R_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) For the row operation  $4R_3$  on matrices with 5 rows, its row operation matrix  $M[4R_3]$  is given by

$$I_5 \xrightarrow{4R_3} M[4R_3] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding ‘reverse row operation’ on matrices with 5 rows is  $\frac{1}{4}R_3$ , and its row operation matrix  $M[\frac{1}{4}R_3]$  is given by

$$I_5 \xrightarrow{\frac{1}{4}R_3} M[\frac{1}{4}R_3] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) For the row operation  $R_2 \leftrightarrow R_5$  on matrices with 5 rows, its row operation matrix  $M[R_2 \leftrightarrow R_5]$  is given by

$$I_5 \xrightarrow{R_2 \leftrightarrow R_5} M[R_2 \leftrightarrow R_5] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding ‘reverse row operation’ on matrices with 5 rows is  $R_2 \leftrightarrow R_5$  itself, and its row operation matrix is  $M[R_2 \leftrightarrow R_5]$  itself.

18. Because of the ‘dictionary’ between row operations and row-operation matrices (Theorem (3)):

- whatever can be done through the application of row operations can also be done through the multiplication from the left by their corresponding row-operation matrices, and
- whatever can be done through the multiplication from the left by row-operation matrices can also be done through the application of the corresponding row operations which determine these row-operation matrices.

For this reason, you may wonder why we need to learn both.

It will transpire that dependent on the situation, to think (and work) with one of them may be more advantageous.

Below is an illustration of a problem which is easy to handle in terms of row operation matrices (and equalities amongst matrices), but not so easy in terms of row operations.

**Illustration.**

$$\text{Let } C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 0 \\ 1 & 3 & 2 \end{bmatrix}.$$

We verify that  $C$  is not row-equivalent to  $D$ :

Suppose it were true that  $C$  was row-equivalent to  $D$ .

Then there would be some sequence of row operations joining  $C$  to  $D$ :

$$C \xrightarrow{\rho_1} \xrightarrow{\rho_2} \dots \xrightarrow{\rho_{k-1}} \xrightarrow{\rho_k} D.$$

Denote the row-operation matrix of  $\rho_j$  by  $M[\rho_j]$  for each  $j$ .

Then the equality

$$D = M[\rho_k]M[\rho_{k-1}] \cdot \dots \cdot M[\rho_2]M[\rho_1]C$$

would hold.

Write  $H = M[\rho_k]M[\rho_{k-1}] \cdot \dots \cdot M[\rho_2]M[\rho_1]$ .

For each  $\ell$ , denote the  $\ell$ -th column of  $C, D$  by  $\mathbf{c}_\ell, \mathbf{d}_\ell$  respectively. Then the equalities

$$\mathbf{d}_1 = H\mathbf{c}_1, \quad \mathbf{d}_2 = H\mathbf{c}_2$$

would hold.

Note that  $\mathbf{c}_2 = 2\mathbf{c}_1$ . Then

$$\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} = \mathbf{d}_2 = H\mathbf{c}_2 = H(2\mathbf{c}_1) = 2H\mathbf{c}_1 = 2\mathbf{d}_1 = 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix},$$

which is impossible.

Hence, in the first place,  $C$  is not row-equivalent to  $D$ .

19. **Theorem (6).** (Re-formulation of row-equivalence between matrices in terms of equalities for matrices.)

Suppose  $A, B$  are matrices with  $p$  rows. Then the statements below are logically equivalent:—

- (1)  $A, B$  are row-equivalent to each other.
- (2) There are (finitely many) row-operation matrices  $G_1, G_2, \dots, G_{k-1}, G_k$  such that  $B = G_k G_{k-1} \cdot \dots \cdot G_2 G_1 A$ .

Now suppose any one of the above holds (so that both hold). Then there are some row-operation matrices  $H_1, H_2, \dots, H_{k-1}, H_k$  such that:—

- $A = H_1 H_2 \cdot \dots \cdot H_{k-1} H_k B$ , and
- for each  $j = 1, 2, \dots, k$ , the equalities  $H_j G_j = I_p$  and  $G_j H_j = I_p$ .

**Proof of Theorem (6).** Exercise. (Apply Theorem (3) and Theorem (5).)

**Remark.** This result is a useful device for theoretical discussions in the future, because we can make use of it to introduce equalities (instead of just row operations) in such discussions.

20. A seemingly trivial consequence of Theorem (6) is that:—

- if a square matrix is row-equivalent to the identity matrix then the square matrix concerned is a product of row-operation matrices.

In fact, much more can be said when we combine what is (implicitly) stated in Theorem (4) about products of row-operation matrices. This will become relevant when we introduce the notion of *invertibility* of square matrices.

**Theorem (7).**

Suppose  $A$  is a  $(p \times p)$ -square matrix. Then the statements below are logically equivalent:—

- (1)  $A$  is row-equivalent to  $I_p$ .
- (2)  $A$  is a product of row-operation matrices.

21. **Proof of Theorem (7).**

Suppose  $A$  is a  $(p \times p)$ -square matrix.

- (a) By Theorem (6), if  $A$  is row-equivalent to  $I_p$  then  $A$  is a product of row-operation matrices.
- (b) Suppose that  $A$  is a product of row-operation matrices, say,  $H_1, H_2, \dots, H_{k-1}, H_k$ , and that the equality  $A = H_1 H_2 \cdots H_{k-1} H_k$  holds.

Denote by  $\gamma_1, \gamma_2, \dots, \gamma_{k-1}, \gamma_k$  the respective row operations on  $(p \times p)$ -matrices to which the row-operation matrices  $H_1, H_2, \dots, H_{k-1}, H_k$  are associated.

We have  $A = H_1 H_2 \cdots H_{k-1} H_k = H_1 H_2 \cdots H_{k-1} H_k I_p$ .

Then we have the sequence of row operations

$$I_p \xrightarrow{\gamma_k} H_k \xrightarrow{\gamma_{k-1}} H_{k-1} H_k \xrightarrow{\gamma_{k-2}} \dots \xrightarrow{\gamma_3} H_3 \cdots H_{k-1} H_k \xrightarrow{\gamma_2} H_2 H_3 \cdots H_{k-1} H_k \xrightarrow{\gamma_1} H_1 H_2 H_3 \cdots H_{k-1} H_k = A$$

Therefore  $I_p$  is row-equivalent to  $A$ . Hence  $A$  is row-equivalent to  $I_p$ .