1.7.1 Exercise: Row operations on matrices.

1. Consider the sequence of row operations below:—

$$A_1 = \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\rho_1} A_2 = \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\rho_2} A_3 = \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\rho_3} A_4 = \begin{bmatrix} 1 & 0 & -3 & -1 & -2 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{\rho_4} A_5 = \begin{bmatrix} 1 & 0 & -3 & -1 & -2 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{\rho_5} A_6 = \begin{bmatrix} 2 & 0 & -6 & -2 & -4 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{\rho_6} A_7 = \begin{bmatrix} 2 & 0 & -6 & -2 & -4 & 0 \\ 0 & 4 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{\rho_7} A_8 = \begin{bmatrix} 2 & 0 & 0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & 4 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{\rho_8} A_9 = \begin{bmatrix} 2 & 0 & 0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & -1 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{bmatrix}$$

   (a) Identify the row operations $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$.

   (b) Identify the row operation ρ_6, and the numbers $\alpha_1, \alpha_2, \alpha_3$.

   (c) Identify the row operation ρ_7, and the numbers $\beta_1, \beta_2, \beta_3$.

   (d) Identify the row operation ρ_8, and the numbers $\gamma_1, \gamma_2, \gamma_3$.

2. For the matrices labelled A, B in each part below, write down an appropriate sequence of row operations which starts at B and ends at A.

   (a) Here A, B are (3×4)-matrices, and are row-equivalent under the sequence of row operations:

$$A \xrightarrow{-1R_1+R_2} \xrightarrow{-2R_1+R_3} \xrightarrow{-2R_2+R_3} \xrightarrow{-1R_3} \xrightarrow{-2R_2+R_1} \xrightarrow{-1R_3+R_2} B$$

   (b) Here A, B are (3×4)-matrices, and are row-equivalent under the sequence of row operations:

$$A \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{-1R_1} \xrightarrow{-2R_1+R_3} \xrightarrow{-3R_2+R_3} \xrightarrow{-2R_2+R_1} B$$

   (c) Here A, B are (3×6)-matrices, and are row-equivalent under the sequence of row operations:

$$A \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{2R_1+R_3} \xrightarrow{-3R_2+R_3} \xrightarrow{-2R_2+R_1} \xrightarrow{2R_3+R_1} \xrightarrow{-2R_3+R_2} B$$

   (d) Here A, B are (4×5)-matrices, and are row-equivalent under the sequence of row operations:

$$A \xrightarrow{-1R_1+R_2} \xrightarrow{-3R_1+R_3} \xrightarrow{-1R_1+R_4} \xrightarrow{-1R_2} \xrightarrow{4R_2+R_3} \xrightarrow{3R_2+R_4} \xrightarrow{R_3 \leftrightarrow R_4} \xrightarrow{4R_3+R_4} \xrightarrow{-2R_2+R_1} \xrightarrow{-1R_3+R_1} B$$

3. (a) Prove the statements below:—

      i. *Suppose C is a $(p \times q)$-matrix. Then C is row-equivalent to C.*

      ii. *Let C, D be $(p \times q)$-matrices. Suppose C is row-equivalent to D. Then D is row-equivalent to C.*

      iii. *Let C, D, E be $(p \times q)$-matrices. Suppose C is row-equivalent to D and D is row-equivalent to E. Then C is row-equivalent to E.*

   (b) Apply mathematical induction to prove the statement below:—
   Let $C_1, C_2, \cdots, C_n, C_{n+1}$ be $(p \times q)$-matrices. Suppose that C_j is row-equivalent to C_{j+1} for each $j = 1, 2, \cdots, n-1, n$. Then C_1 is row-equivalent to C_n.

4. (a) Let A, A' be $(m \times p)$-matrices, and B be a $(p \times q)$-matrix.
   Suppose A' is resultant from the application of one row operation ρ on A.
   Prove the statements below, with direct reference to the definitions of the respective types of row operations:—

      i. *Suppose ρ is given by $\alpha R_i + R_k$ for some number $\alpha$and some distinct integers i, k between 1 and m. Then AB is resultant from the application of ρ on $A'B$.*

      ii. *Suppose ρ is given by βR_k for some non-zero number β and some integer k. Then AB is resultant from the application of ρ on $A'B$.*

      iii. *Suppose ρ is given by $R_i \leftrightarrow R_k$ for some integers i, k. Then AB is result from the application of ρ on $A'B$.*

(b) Apply mathematical induction to prove the statement below, with reference to the properties of row operations.

Suppose A, A' are row-equivalent $(m \times p)$-matrices. Then $AB, A'B$ are row-equivalent for any $(p \times q)$-matrix B.

Remark. An appropriate proposition $P(n)$ upon which mathematical induction is applied is given by:—

Suppose A, A' are row-equivalent $(m \times p)$-matrices under some sequence of n row operations:

$$A \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_n} A'.$$

Then $AB, A'B$ are row-equivalent under the same sequence of row operations for any $(p \times q)$-matrix B:

$$AB \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_n} A'B.$$