1.7 Row operations on matrices.

0. *Assumed background.*

   - 1.1 *Matrices, matrix addition, and scalar multiplication for matrices.*

   Abstract. We introduce:—

   - the notion of row operations on matrices,
   - the notions of 'reverse' row operations, and sequences of row operations,
   - the notion of row equivalence.

1. **Definition. (Row operation 'adding a scalar multiple of one row to another'.)**

   Let A be a $(p \times q)$-matrix whose (i,j)-th entry is denoted by a_{ij}, and whose k-th row is denoted by $\mathbf{a}_k$.

   Suppose α is a number.

   When we replace the k-th row $[\ a_{k1} \quad a_{k2} \quad \cdots \quad a_{kq}\ ]$ of A by

   $$[\ \alpha a_{i1} + a_{k1} \quad \alpha a_{i2} + a_{k2} \quad \cdots \quad \alpha a_{iq} + a_{kq}\ ],$$

   in which $i \neq k$, to obtain some resultant matrix A', we say we are applying the row operation '$\alpha \cdot R_i + R_k$' to A, and write $A \xrightarrow{\alpha R_i + R_k} A'$.

   Such a row operation is called **adding a scalar multiple of one row of A to another row of A.**

2. **Example (1). (Adding a scalar multiple of one row to another.)**

   (a) $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1 + R_2} A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$

   (b) $A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2 + R_1} A'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$

3. **Definition. (Row operation 'multiplying a non-zero scalar to a row'.)**

   Let A be a $(p \times q)$-matrix whose (i,j)-th entry is denoted by a_{ij}, and whose k-th row is denoted by $\mathbf{a}_k$.

   Suppose β is a non-zero number.

   When we replace the k-th row $[\ a_{k1} \quad a_{k2} \quad \cdots \quad a_{kq}\ ]$ of A by

   $$[\ \beta a_{k1} \quad \beta a_{k2} \quad \cdots \quad \beta a_{kq}\ ]$$

   to obtain some resultant matrix A', we say we are applying the row operation '$\beta \cdot R_k$' to A, and write $A \xrightarrow{\beta R_k} A'$.

   Such a row operation is called **multiplying a non-zero scalar to a row of A.**

4. **Example (2). (Multiplying a non-zero scalar to a row.)**

   (a) $B = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} B' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$

   (b) $B' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} B'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$

5. **Definition. (Row operation 'interchanging two rows'.)**

   Let A be a $(p \times q)$-matrix whose (i,j)-th entry is denoted by a_{ij}, and whose k-th row is denoted by $\mathbf{a}_k$.

   When we interchange the i-th row $[\ a_{i1} \quad a_{i2} \quad \cdots \quad a_{iq}\ ]$ and the k-th row $[\ a_{k1} \quad a_{k2} \quad \cdots \quad a_{kq}\ ]$ of A, in which $i \neq k$, to obtain some resultant matrix A', we say we are applying the row operation '$R_i \longleftrightarrow R_k$' to A, and write $A \xrightarrow{R_i \leftrightarrow R_k} A'$.

   Such a row operation is called **interchanging two rows of A.**

6. **Example (3). (Interchanging two rows.)**

   (a) $C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$

(b) $C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}$.

7. **Definition. (Row operations.)**

   Let A, A' be $(p \times q)$-matrices.

   We say we are **applying a row operation** *on A to obtain A' if and only if A' is the resultant of the application of*

   - *one row operation adding a scalar multiple of one row of A to another row of A, or*
   - *one row operation multiplying a non-zero scalar to a row of A, or*
   - *one row operation interchanging two rows of A.*

8. **Definition. (Sequences of row operations.)**

   Let $A_1, A_2, \cdots, A_{N-1}, A_N$ be finitely many $(p \times q)$-matrices.

   Suppose that for each k, A_{k+1} is the resultant of the application of one row operation, say, ρ_k, on A_k. Then we say that $A_1, A_2, \cdots, A_{N-1}, A_N$ is joint by the **sequence of row operations** *$\rho_1, \rho_2, \cdots, \rho_{N-1}$.*

   When we want to emphasize that for each k, the row operation ρ_k is applied to A_k to obtain A_{k+1}, we may present this sequence as

   $$A_1 \xrightarrow{\rho_1} A_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{N-2}} A_{N-1} \xrightarrow{\rho_{N-1}} A_N.$$

   We may also refer to such a sequence as the sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{N-1}$ when we want to emphasize the role of the row operations.

   Remark. When $N = 1$, we have the 'trivial sequence' of row operations 'A_1'.

9. **Example (4). (Sequences of row operations.)**

   (a) $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2+R_1} A'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

   (b) $B = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} B' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} B'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

   (c) $C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}$.

   (d) $A_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} A_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_3} A_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} A_4 = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.

10. We can 'undo' the effect of any row operation on any given matrix and thus 'restore' that given matrix by applying onto the resultant an appropriate row operation onto the resultant.

    Although this looks innocent enough (when presented explicitly in the form of Theorem (1) and its remarks), this is the origin of many deep results in this course.

    Theorem (1). (Existence and uniqueness of 'reverse row operations'.)

    Suppose ρ is a row operation on $(p \times q)$-matrices.

    Then there is some unique row operation $\tilde{\rho}$ on $(p \times q)$-matrices such that the application of the sequence of row operations $\rho, \tilde{\rho}$ onto any $(p \times q)$-matrix A results in A.

    Proof of Theorem (1).

    We will give an outline of the argument later. (A full proof is a tedious but straightforward word game about the definitions.)

11. **Remark on the content of Theorem (1).**

    What we are saying in the conclusion of Theorem (1) is that given any $(p \times q)$-matrices A, A', if

    $$A \xrightarrow{\rho} A'$$

    is valid then:—

(1) there is a corresponding row operation $\widetilde{\rho}$ for which the sequence

$$A \xrightarrow{\rho} A' \xrightarrow{\widetilde{\rho}} A$$

is valid,

(2) such a $\widetilde{\rho}$ is determined by ρ alone (but not by A),

(3) such a $\widetilde{\rho}$ is uniquely determined.

Further remark on terminology.

Theorem (1) justifies the naming of $\widetilde{\rho}$ as the **'reverse row operation' for** ρ.

Dependent on the 'type' to which ρ belongs, $\widetilde{\rho}$ will be given correspondingly, as described by the table below:—

Row operation ρ on A resultant in A'.	Corresponding 'reverse row operation' $\widetilde{\rho}$ on A' resultant in A.
$A \xrightarrow{\alpha R_i + R_k} A'$.	$A' \xrightarrow{-\alpha R_i + R_k} A$.
$A \xrightarrow{\beta R_k} A'$.	$A' \xrightarrow{(1/\beta) R_k} A$.
$A \xrightarrow{R_i \leftrightarrow R_k} A'$.	$A' \xrightarrow{R_i \leftrightarrow R_k} A$.

The entries in this table are justified by the argument for Theorem (1).

From this table, it is apparent that Theorem (2) holds.

12. **Theorem (2). ('Reverse' of 'reverse row operation'.)**

Suppose ρ is a row operation on matrices with p rows, and $\widetilde{\rho}$ is the 'reverse row operation' for ρ.

Then the 'reverse row operation' for $\widetilde{\rho}$ is given by ρ itself.

Remark. In symbols, what we are saying is:—

If

$$A \xrightarrow{\quad\rho\quad}\xrightarrow{\quad\widetilde{\rho}\quad} A$$

is valid for each matrix A with p rows, then

$$B \xrightarrow{\quad\widetilde{\rho}\quad}\xrightarrow{\quad\rho\quad} B$$

is valid for each matrix B with p rows.

13. **Example (5). ('Reverse row operations'.)**

(a) The 'reverse' row operation for

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1 + R_2} A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

is given by

$$A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1R_1 + R_2} A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(b) The 'reverse' row operation for

$$B = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} B' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

is given by

$$B' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{4}R_2} B = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(c) The ‘reverse’ row operation for

$$C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

is given by

$$C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$$

14. We can recover a matrix from the resultant of a sequence of row operations on the matrix by ‘reversing’ the sequence and ‘replacing’ the respective row operations with their ‘reverse’ row operations.

Theorem (3). (‘Reversing’ a sequence of row operations.)

Suppose $A_1, A_2, \cdots, A_N$ is a sequence of $(p \times q)$-matrices joint by row operations $\rho_1, \rho_2, \cdots, \rho_{N-1}$ respectively:

$$A_1 \xrightarrow{\rho_1} A_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{N-2}} A_{N-1} \xrightarrow{\rho_{N-1}} A_N.$$

Then $A_N, \cdots, A_2, A_1$ is a sequence of $(p \times q)$-matrices joint by row operations $\widetilde{\rho_{N-1}}, \cdots, \widetilde{\rho_2}, \widetilde{\rho_1}$ respectively, in which $\widetilde{\rho_k}$ is the ‘reverse row operation’ of ρ_k for each k:

$$A_N \xrightarrow{\widetilde{\rho_{N-1}}} A_{N-1} \xrightarrow{\widetilde{\rho_{N-2}}} \cdots \xrightarrow{\widetilde{\rho_2}} A_2 \xrightarrow{\widetilde{\rho_1}} A_1.$$

Proof of Theorem (3). The argument is a repeated application of Theorem (1).

15. **Example (6). (‘Reversing’ a sequence of row operations.)**

(a) We obtain A'' from A by applying this sequence of row operations:—

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1 + R_2} A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_2 + R_1} A'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

We ‘recover’ A from A'' by ‘reversing’ the above sequence:—

$$A'' = \begin{bmatrix} 3 & 4 & 5 & 5 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_2 + R_1} A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1R_1 + R_2} A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(b) We obtain A'' from A by applying this sequence of row operations:—

$$B = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{4R_2} B' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_1} B'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

We ‘recover’ B from B'' by ‘reversing’ the above sequence:—

$$B'' = \begin{bmatrix} -2 & -4 & -4 & 2 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} B' = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 8 & -8 & 4 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{4}R_2} B = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(c) We obtain C'' from C by applying this sequence of row operations:—

$$C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}.$$

We ‘recover’ C from C'' by ‘reversing’ the above sequence:—

$$C'' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} C' = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$$

(d) We obtain A_4 from A_1 by applying this sequence of row operations:—

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{1R_1+R_2} A_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_3} A_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} A_4 = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

We 'recover' A_1 from A_4 by 'reversing' the above sequence:—

$$A_4 = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} A_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{(1/2)R_3} A_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1R_1+R_2} A_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

16. **Definition. (Row-equivalent matrices.)**

Let C, D be $(p \times q)$-matrices.

Suppose there is a finite sequence of row operations starting from C and ending at D.

Then we say that C is **row-equivalent to** D.

When we also want to emphasize that C is joint to D by, say, some sequence of row operations

$$C \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_k} D,$$

we say that C is **row-equivalent to D under the sequence of row operations** $\rho_1, \rho_2, \cdots, \rho_k$.

17. **Question.** *How to show that a given $(p \times q)$-matrix C is row-equivalent to a $(p \times q)$-matrix D?*

Answer. Write down a finite sequence of $(p \times q)$-matrices, say,

$$C = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{N-2}} C_{N-1} \xrightarrow{\rho_{N-1}} C_N = D$$

joint by row operations, one at each step.

Illustration.

Let $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ *and* $D = \begin{bmatrix} 3 & 6 & 6 \\ 8 & -8 & 4 \end{bmatrix}$.

We verify that C is row-equivalent to D:

$$\begin{aligned} C \;=\; & C_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{1R_1+R_2} C_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow{3R_1} C_3 = \begin{bmatrix} 3 & 0 & 3 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} C_4 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 3 \end{bmatrix} \\ & \xrightarrow{-1R_1+R_2} C_5 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix} \xrightarrow{4R_2} C_6 = \begin{bmatrix} 1 & 2 & 2 \\ 8 & -8 & 4 \end{bmatrix} \xrightarrow{3R_1} C_7 = D = \begin{bmatrix} 3 & 6 & 6 \\ 8 & -8 & 4 \end{bmatrix} \end{aligned}$$

18. **Theorem (4). (Row-equivalence as an 'equivalence relation'.)**

The statements below hold:—

(a) *Suppose C is a $(p \times q)$-matrix. Then C is row-equivalent to C.*

(b) *Let C, D be $(p \times q)$-matrices. Suppose C is row-equivalent to D. Then D is row-equivalent to C.*

(c) *Let C, D, E be $(p \times q)$-matrices. Suppose C is row-equivalent to D and D is row-equivalent to E. Then C is row-equivalent to E.*

Proof of Theorem (4). Exercise.

Remark on the significance of Theorem (4). By virtue of Theorem (4), it will make sense for us to write something like

'*the matrices A, B are row-equivalent to each other*', '*the matrices C, D, E are row-equivalent to each other*',

using the phrase 'row-equivalent' in similar way that we use the word 'equal' (in various areas of mathematics) or the phrase 'congruent' (in plane geometry).

Further remark. According to Theorem (4), the collection of all $(p \times q)$-matrices are split into various 'cliques' according to the question whether one $(p \times q)$-matrix is row-equivalent to another $(p \times q)$-matrix. If yes, then the two matrices concerned are in the same 'clique'; if no, they are not.

19. **Outline of argument for Theorem (1).**

Suppose ρ is a row operation on $(p \times q)$-matrices.

(a) We first argue for the existence of an appropriate row operation, labeled $\widetilde{\rho}$, on $(p \times q)$-matrices for which the sequence of row operations

$$A \xrightarrow{\quad\rho\quad}\xrightarrow{\quad\widetilde{\rho}\quad} A$$

is valid.
Note that ρ is given by one of

$$\alpha R_i + R_k, \qquad \beta R_k \qquad R_i \leftrightarrow R_k$$

for some appropriate α, β, i, k.
Respectively introduce the row operation $\tilde{\rho}$ to be one of

$$-\alpha R_i + R_k, \qquad \beta^{-1} R_k \qquad R_i \leftrightarrow R_k$$

for the same α, β, i, k.
Then in each case, for any $(p \times q)$-matrix A, in the sequence of row operations

$$A \xrightarrow{\rho} A' \xrightarrow{\widetilde{\rho}} A'',$$

we will obtain $A'' = A$.
[To see this we have to check according to definition that the sequences of row operations below are indeed valid:—

- $A \xrightarrow{\alpha R_i + R_k} A' \xrightarrow{-\alpha R_i + R_k} A$.
- $A \xrightarrow{\beta R_k} A' \xrightarrow{\beta^{-1} R_k} A$.
- $A \xrightarrow{R_i \leftrightarrow R_k} A' \xrightarrow{R_i \leftrightarrow R_k} A$.

This is left as an exercise.]

This completes the argument for the 'existence part'.

(b) Now we turn to the argument for 'uniqueness part'.

Suppose $\hat{\rho}$ is a row operation on $(p \times q)$-matrices for which the successive application of $\rho, \hat{\rho}$ on each $(p \times q)$-matrix A results in A.
It suffices for us to justify these claims:—

- If ρ is given by $\alpha R_i + R_k$ for some number α, then $\hat{\rho}$ is given by $-\alpha R_i + R_k$.
- If ρ is given by βR_k for some nonzero number β, then $\hat{\rho}$ is given by $\beta^{-1} R_k$.
- If ρ is given by $R_i \leftrightarrow R_k$ then $\hat{\rho}$ is given by $R_i \leftrightarrow R_k$.

(For each claim, we can give an easy argument by choosing A judiciously in the argument. An obvious choice is '$A = I_p$'.)