1.6.1 Linear dependence and linear independence.

1. With direct reference to the definitions of linear dependence, linear independence, and linear combinations (where relevant), prove the statements below:—

   (a) $\mathbf{0}_p$ *is linearly dependent.*

   (b) *Suppose* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are column vectors with* p *entries.*
   Then $\mathbf{0}_p, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly dependent.*

   (c) *Suppose* $\mathbf{u}$ *is a column vector with* p *entries.*
   Then, for any non-zero numbers α, β*, the column vectors* $\alpha\mathbf{u}, \beta\mathbf{u}$ *are linearly dependent.*

   (d) *Suppose* $\mathbf{u}, \mathbf{v}$ *are column vector with* p *entries.*
   Then the column vectors $\mathbf{u}, \mathbf{v}, \mathbf{u} - \mathbf{v}$ *are linearly dependent.*

   (e) *Suppose* $\mathbf{u}, \mathbf{v}$ *are column vector with* p *entries.*
   Then, for any numbers $\alpha, \beta, \gamma, \delta$*, the column vectors* $\alpha\mathbf{u}, \beta\mathbf{v}, \gamma\mathbf{u} + \delta\mathbf{v}$ *are linearly dependent.*
   Remark. The argument requires some care. Just as you cannot expect a symbol like $\mathbf{u}$ to automatically stand for a non-zero column/row vector, you cannot expect a symbol like α to automatically stand for a non-zero number.

   (f) *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}$ *be column vector with* p *entries. Suppose the column vectors* $\mathbf{v}, \mathbf{w}, \mathbf{x}$ *are linearly dependent.*
   Then the column vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}$ *are linearly dependent.*

2. Let $\mathbf{u}, \mathbf{v}$ be column vectors with p entries. Prove, with direct reference to the definitions for the notions of linear dependence and linear combination, that the statements below are logically equivalent:—

   (1) $\mathbf{u}, \mathbf{v}$ are linearly dependent.

   (2) One of $\mathbf{u}, \mathbf{v}$ is a scalar multiple of the other.

3. Let $\mathbf{u}, \mathbf{v}$ be column vectors with p entries. Prove that the statements below are logically equivalent, with direct reference to the definition of linear independence:—

   (1) $\mathbf{u}, \mathbf{v}$ *are linearly independent.*

   (2) $\mathbf{u}, -\mathbf{v}$ *are linearly independent.*

   (3) $\mathbf{u} + \mathbf{v}, \mathbf{u}$ *are linearly independent.*

   (4) $\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}$ *are linearly independent.*

4. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be column vectors with p entries. Prove that the statements below are logically equivalent, with direct reference to the definition of linear dependence:—

   (1) $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are linearly dependent.*

   (2) $\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}$ *are linearly dependent.*

   (3) $\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{w}$ *are linearly dependent.*

   (3) $\mathbf{u} + \mathbf{v} + \mathbf{w}, \mathbf{u} + 2\mathbf{v} + 3\mathbf{w}, \mathbf{u} + 3\mathbf{v} + 6\mathbf{w}$ *are linearly dependent.*

5. (a) Prove the statement (♯):—

   (♯) *Let* $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ *be column vectors with* p *entries. Suppose* $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ *are linearly independent.*
   Then $\mathbf{u} + 2\mathbf{t}, \mathbf{v} + 3\mathbf{t}, \mathbf{w} + 4\mathbf{t}$ *are linearly independent.*

   (b) i. Give a dis-proof against (♭) by providing an appropriate counter-example.

   (♭) *Let* $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ *be column vectors with* 3 *entries. Suppose* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are linearly independent.*
   Then $\mathbf{u} + 2\mathbf{t}, \mathbf{v} + 3\mathbf{t}, \mathbf{w} + 4\mathbf{t}$ *are linearly independent.*

   ii. *This is more challenging than the previous part.*
   Give a dis-proof against (♭′) by providing an appropriate counter-example.

   (♭′) *Let* $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ *be column vectors with* 3 *entries. Suppose* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are linearly independent, and none of* $\mathbf{u} + 2\mathbf{t}, \mathbf{v} + 3\mathbf{t}, \mathbf{w} + 4\mathbf{t}$ *is the zero column vector.*
   Then $\mathbf{u} + 2\mathbf{t}, \mathbf{v} + 3\mathbf{t}, \mathbf{w} + 4\mathbf{t}$ *are linearly independent.*

6. With direct reference to the definitions of linear dependence, linear independence, and linear combinations (where relevant), prove the statements below.

(a) *Let* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *be column vectors with* p *entries. Suppose* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *are linearly independent. Then for each* $j = 1, 2, \cdots, k$*, the column vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_j$ *are linearly independent.*

(b) *Let* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_q$ *be column vectors with* p *entries.*
Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ *are linearly dependent.*
Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_q$ *are linearly dependent.*

Remark. In plain words, the statement in part (a) says:—

Any 'portion' of a collection of linearly independent column vectors is itself a collection of linearly independent column vectors.

The statement in part (b) says:—

A collection of column vectors is definitely linearly dependent when some portion of it is, on its own, a collection of linearly dependent column vectors.

The two statements are 'contra-positive' re-formulations of each other.

7. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be column vectors with m entries.
Let $\mathbf{v}_1 = 2\mathbf{u}_1 + \mathbf{u}_2 - 3\mathbf{u}_3$, $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{v}_3 = \mathbf{u}_2 + \mathbf{u}_3$.

(a) Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.
Let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ be numbers.

i. Suppose $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \beta\mathbf{u}_3$.
Express $\beta_1, \beta_2, \beta_3$ in terms of $\alpha_1, \alpha_2, \alpha_3$. (Justify your answer.)

ii. Are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly independent?
Hint. Start with the assumption '*suppose* $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$'. What can you say about the values of $\alpha_1, \alpha_2, \alpha_3$?

Remark. The assumption '$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linear independent* is crucial in guaranteeing the conclusion '$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *are linearly independent*'.

To explore, write down any concrete example of non-zero column vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, say, 3 entries, which are linear dependent. See whether the corresponding column vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ will automatically turn out to be linearly dependent.

(b) i. Express each of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ as linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
Hint. Start by expressing $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_2 + \mathbf{v}_3$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

ii. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. Show that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is linearly independent.

Remark. So it happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is linearly independent if and only if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent.