1.5.1 Exercise: Linear combinations.

In the questions below, unless otherwise stated, the words 'numbers', 'scalars', 'entries' ... should be understood respectively as 'real numbers', 'real scalars', 'real entries' ...

That said, when the words 'numbers', 'scalars', 'entries' ... in the 'theory-type' questions are consistently read as 'complex numbers', 'complex scalars', 'complex entries', the results these questions describe are also valid.

1. Let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

   (a) Suppose $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ with respect to scalars $1, 3, 1, 0, 2$.
   Find $\mathbf{v}$, (giving its entries explicitly).

   (b) Let $\mathbf{w} = \begin{bmatrix} 2 \\ 8 \\ 7 \\ 10 \end{bmatrix}$.
   Suppose $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ with respect to scalars $1, -2, 2, -1, \alpha$, in which α is some number.
   Find the value of α.

2. Let a be a real number, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ a \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ a \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} a^2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -a \\ 2a^2 \\ a^2 \\ a^3 \end{bmatrix}$.

   Suppose $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ with respect to the scalars $a, 3, -1$.

   (a) Find the value of a.

   (b) Show that $\mathbf{u}_1 + \mathbf{u}_2$ is a linear combination of $\mathbf{u}_2, \mathbf{u}_3, \mathbf{v}$. Write down a linear relation relating $\mathbf{u}_1 + \mathbf{u}_2$ with $\mathbf{u}_2, \mathbf{u}_3, \mathbf{v}$, giving the scalars involved explicitly.

3. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$ be row vectors with n entries.
   Prove the statements below, with direct reference to the relevant definitions:—

   (a) *The zero row vector* $\mathbf{0}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$.

   (b) *Let* $\mathbf{v}, \mathbf{w}$ *be row vectors with* n *entries. Suppose each of* $\mathbf{v}, \mathbf{w}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$. *Then* $\mathbf{v} + \mathbf{w}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$.

   (c) *Let* $\mathbf{y}$ *be a row vector with* n *entries, and* α *be a scalar. Suppose* $\mathbf{y}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$. *Then* $\alpha\mathbf{y}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$.

4. Let $\mathbf{v}, \mathbf{w}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be column vectors with m entries.
   Suppose $\mathbf{v} = 2\mathbf{u}_1 + 3\mathbf{u}_2 - 4\mathbf{u}_3$, and $\mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2$.

   (a) With direct reference to the relevant definitions, show that $\mathbf{u}_1$ is a linear combination of $\mathbf{v}, \mathbf{u}_2, \mathbf{u}_3$.

   (b) i. With direct reference to the relevant definitions, show that $a\mathbf{v} + b\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ for every number a, b.
   ii. By applying the results in your calculations in part (b.i), or otherwise, show that each of $\mathbf{u}_1, \mathbf{u}_2$ is a linear combination of $\mathbf{v}, \mathbf{w}, \mathbf{u}_3$.

5. (a) Let $\mathbf{v}$, $\mathbf{u}$, $\mathbf{w}$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$, $\mathbf{y}_2$ be column vectors with m entries.
   Suppose $\mathbf{u} = 3\mathbf{x}_1 - 2\mathbf{x}_2$, $\mathbf{w} = 2\mathbf{y}_1 - \mathbf{y}_2$ and $\mathbf{v} = 3\mathbf{u} - \mathbf{w}$.
   With direct reference to the relevant definitions, show that $\mathbf{v}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1$ and $\mathbf{y}_2$.

   (b) Let $\mathbf{u}$, $\mathbf{w}$, $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k, \mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_h$ be column vectors with m entries.
   Suppose $\mathbf{u}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$, and $\mathbf{w}$ is a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_h$.
   Show that $\alpha\mathbf{u} + \beta\mathbf{w}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k, \mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_h$ for any numbers α, β.

6. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}$ be column vectors with 5 entries.

   (a) Prove the statements below:—

i. *Suppose* $\mathbf{v}$ *is a linear combination of* $\frac{1}{\sqrt{2}}(\mathbf{u}_1 + \mathbf{u}_2), \frac{1}{\sqrt{2}}(\mathbf{u}_1 - \mathbf{u}_2)$*. Then* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2$*.*

ii. *Suppose* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2$*. Then* $\mathbf{v}$ *is a linear combination of* $\frac{1}{\sqrt{2}}(\mathbf{u}_1 + \mathbf{u}_2), \frac{1}{\sqrt{2}}(\mathbf{u}_1 - \mathbf{u}_2)$*.*

(b) Suppose a, b are numbers, and suppose $a^2 + b^2 = 1$.

Prove the statement ($\sharp$):

($\sharp$) $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2$ *if and only if* $\mathbf{v}$ *is a linear combination of* $a\mathbf{u}_1 + b\mathbf{u}_2, -b\mathbf{u}_1 + a\mathbf{v}_2$*.*

7. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}$ be column vectors with 8 entries.

Suppose that the column vector $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Further suppose that for each $j = 1, 2, 3, 4, 5$, the column vector $\mathbf{v}_j$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$.

Show that $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$, with direct reference to the relevant definitions.