1.4.3 Exercise: Commuting matrices versus non-commuting matrices.

In some of the questions below, you will need the respective notions of *diagonal matrix*, *polynomial of a square matrix*, *Lie product*. Their respective definitions are given below:—

- *Let D be a $(p \times p)$-square matrix, whose (i, j)-th entry is denoted by d_{ij} for each i, j.*

  We say that D is a **diagonal matrix** *if and only if $d_{ij} = 0$ whenever $i \neq j$.*

  We may write $D = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_p)$ if $d_{11} = \alpha_1$, $d_{22} = \alpha_2$, ... and $d_{pp} = \alpha_p$. In this situation, we may further call D the diagonal matrix with respective diagonal entries $\alpha_1, \alpha_2, \cdots, \alpha_p$.

- *Suppose B is a $(p \times p)$-square matrix, and $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ be numbers.*

  Then the $(p \times p)$-square matrix given by

  $$a_0 I_p + a_1 B + a_2 B^2 + \cdots + a_{n-1} B^{n-1} + a_n B^n$$

  is called the **polynomial of B with respective coefficients** $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$.

  For convenience of notations, if $f(x)$ is the polynomial with variable x given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$, then we agree to write

  $$f(B) = a_0 I_p + a_1 B + a_2 B^2 + \cdots + a_{n-1} B^{n-1} + a_n B^n.$$

- *Let A, B be square matrices of the same size.*

  The square matrix $AB - BA$ is called the **Lie product of** *A, B, and is denoted by $[A, B]$.*

  Remark. $[A, B]$ 'measures' how far AB and BA differ from each other.

1. Let $A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & -2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

   Do A, B commute with each other? Justify your answer.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

   Do A, B commute with each other? Justify your answer.

3. (a) Let A, B be square matrices with size m. Suppose $AB = BA = \mathcal{O}$.

   i. Verify that $(A + B)^2 = A^2 + B^2$.

   ii. Verify that $(A + B)^3 = A^3 + B^3$.

   (b) Without using the 'Binomial Theorem for commuting matrices', prove the statement below, with the help of mathematical induction:—
   Let A, B be square matrices with size m. Suppose $AB = BA = \mathcal{O}$. Then for any integer n greater than 1, the equality $(A + B)^n = A^n + B^n$.

   (c) Let $A = \begin{bmatrix} 2 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -4 & -2 \\ 6 & 6 & 0 \\ -3 & -2 & 2 \end{bmatrix}$.

   Show that $(A + B)^n = A^n + B^n$ for each integer $n \geq 2$.
   Hint. Compute AB and BA first.

4. (a) Prove the statement (♯):—
   Suppose D, E are diagonal matrices of the same size. Then D, E commute with each other.

   (b) Prove the statement (♮):—
   Let A, B, G, H be $(p \times p)$-square matrices. Suppose GAH, GBH are diagonal matrices and $HG = I_p$. Then A, B commute with each other.

5. (a) Prove the statement below (with the help of mathematical induction, if appropriate):—
   Let A, B be $(p \times p)$-square matrix. Suppose A, B commute with each other. Then, for any positive integers m, n, the matrices A^m, B^n commute with each other.
   Hint. This statement can be re-formulated as:—

Let A, B be $(p \times p)$-square matrix. Suppose A, B commute with each other. Suppose n is a positive integer. Then, for any positive integer m, the matrices A^m, B^n commute with each other.

Remark. You may take for granted the validity of the statement below:—

Let C, D be $(p \times p)$-square matrix. Suppose C, D commute with each other. Then, for any positive integer n, the matrices C, D^n commute with each other.

(b) Prove the statement below (with the help of mathematical induction, if appropriate):—
Let A, B be $(p \times p)$-square matrix. Suppose A, B commute with each other. Then, for any positive integer n, the equality

$$(A+B)^n = A^n + \binom{n}{1} A^{n-1}B + \binom{n}{2} A^{n-2}B^2 + \cdots + \binom{n}{k} A^{n-k}B^k + \cdots + \binom{n}{n-1} AB^{n-1} + B^n$$

holds .

6. (a) Prove the statement below (with the help of mathematical induction, if appropriate):—
Let A, B be $(p \times p)$-square matrices. Suppose A, B commute with each other. Then, for any positive integer n, the equality

$$A^{n+1} - B^{n+1} = (A-B)(A^n + A^{n-1}B + A^{n-2}B^2 + \cdots + A^{n-k}B^k + \cdots + AB^{n-1} + B^n)$$

holds .

(b) Hence, or otherwise, deduce the statement below:—
Let B be a $(p \times p)$-square matrix. Then, for any positive integer n,

$$I_p - B^{n+1} = (I_p - B)(I_p + B + B^2 + \cdots + B^k + \cdots + B^{n-1} + B^n)$$

7. (a) Prove the statement ($\sharp$):—
Let B, C be square matrices of the same size, and $f(x)$ be a polynomial.
Suppose B, C commute with each other. Then $f(B), C^m$ commute with each other for each positive integer m.
Remark. You may take for granted that under the assumption that B, C commute with each other, it happens that B^k, C^m will commute with each other for any positive integers k, m.

(b) Hence, or otherwise, prove the statement ($\sharp\sharp$):—
Let B, C be square matrices of the same size, and $f(x), g(x)$ be polynomials.
Suppose B, C commute with each other. Then $f(B), g(C)$ commute with each other.

8. Let $J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.
Verify that $[J, K] = L$, $[K, L] = J$, $[L, J] = K$.

9. Prove the statements below:—

(a) *Suppose A, B, C are square matrices of the same size, and β, γ are numbers. Then $[A, \beta B + \gamma C] = \beta[A, B] + \gamma[A, C]$.*

(b) *Suppose A is a square matrix. Then $[A, A] = \mathcal{O}$.*

(c) *Suppose A, B are square matrices of the same size. Then $[A, B] = -[B, A] = [-B, A] = [B, -A]$.*

(d) *Suppose A, B, C are square matrices of the same size, and α, β are numbers. Then $[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C]$.*

(e) *Let A, B be square matrices of the same size. Suppose $A, [A, B]$ commute with each other. Then $[A^{k+1}, B] = (k+1)[A, B]A^k$ for each positive integer k.*

10. Prove the statements below:—

(a) *Suppose A, B, C are square matrices of the same size. Then $[\ A, [B, C]\ ] + [\ B, [C, A]\ ] + [\ C, [A, B]\ ] = \mathcal{O}$.*

(b) *Suppose A, B are $(n \times n)$-square matrices. Further suppose that A, B commute with each other. Then $[\ A, [B, C]\ ] = [\ B, [A, C]\ ]$ for any $(n \times n)$-square matrix C.*

(c) *Suppose A, B, C, D are square matrices of the same size. Then $[\ A, [\ B, [C, D]\ ]\ ] + [\ B, [\ C, [D, A]\ ]\ ] + [\ C, [\ D, [A, B]\ ]\ ] + [\ D, [\ A, [B, C]\ ]\ ] = \mathcal{O}$.*

11. Prove the statements below:—

(a) *Let A be a square matrix. Suppose $B = [A, A^t]$. Then B is symmetric.*

(b) *Let A, B be square matrices of the same size. Suppose A, B are symmetric. Then $[A, B]$ is skew-symmetric.*

(c) *Let A, B be square matrices of the same size. Suppose A is symmetric and B is skew-symmetric. Then $[A, B]$ is symmetric.*

(d) *Let A, B be square matrices of the same size. Suppose A, B are skew-symmetric. Then $[A, B]$ is skew-symmetric.*

12. (a) Let A, B be square matrices.

Express each of $(A + B)^2$, $(A - B)(A + B)$ in the form of $\alpha A^2 + \beta B^2 + \gamma AB + \delta[A, B]$, in which $\alpha, \beta, \gamma, \delta$ are appropriate numbers.

(b) Hence, or otherwise, prove that the statement below:—

Suppose A, B are square matrices.

Then $(A + B)^2 = A^2 + 2AB + B^2$ if and only if $(A - B)(A + B) = A^2 - B^2$.

13. Prove the statement below:—

Let A, B, C, D be square matrices of the same size, and s, t, u, v be numbers.

Suppose $C = sA + tB$, $D = uA + vB$, and $sv - tu \neq 0$.

Then $[A, B] = \mathcal{O}$ if and only if $[C, D] = \mathcal{O}$.