1.4.2 Appendix: Dis-proof by providing a counter-example.

0. The material in this appendix is supplementary.

1. In this course we have already encountered (and will encounter) many a statement which is of the form as $(\star)$:—

   $(\star)$ '**Let/suppose** *so-and-so* **be/is amongst the objects** *tum-tum-tum-tum-tum-tum.*
   Suppose *so-and-so possesses the property*

   $$\underbrace{\textit{blah-blah-blah-blah-blah-blah-blah-blah-blah}}_{\text{a statement about } \textit{so-and-so} \text{ when read on its own}} .$$

   Then *so-and-so possesses the property*

   $$\underbrace{\textit{bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh}}_{\text{a statement about } \textit{so-and-so} \text{ when read on its own}} .$$

   ,

   Or we may encounter the *same* statement in the disguise of $(\star')$:

   $(\star')$ '**For any** *so-and-so* **amongst the objects** *tum-tum-tum-tum-tum-tum,* **if** *so-and-so possesses the property*

   $$\underbrace{\textit{blah-blah-blah-blah-blah-blah-blah-blah-blah}}_{\text{a statement about } \textit{so-and-so} \text{ when read on its own}}$$

   then *so-and-so possesses the property*

   $$\underbrace{\textit{bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh}}_{\text{a statement about } \textit{so-and-so} \text{ when read on its own}}$$

   ,

2. Not every statement of the likes of $(\star)$ is true.

   When we claim that such a statement $(\star)$ is false, and we want to rigorously explain this claim in in a mathematical argument, we give an argument that is known as **dis-proof by providing a counter-example** against $(\star)$.

   The work for such an argument consists of:—

   (0) *Important preparatory step, but not part of the argument.*
   Conceive (or make an **educated guess** on) a **concrete** object *so-and-so*:—
   (0a) which we *believe/hope* will be amongst the objects *tum-tum-tum-tum-tum-tum*,
   (0b) which we also *believe/hope* will possess the property *blah-blah-blah-blah-blah-blah*, and
   (0c) which we further *believe/hope* will fail to possess the property *bleh-bleh-bleh-bleh-bleh-bleh.*

   (1) *First step of the argument.*
   Unveil the concrete object *so-and-so.*

   (2) *Rest of the argument.*
   Verify that:—
   (2a) *so-and-so* is indeed amongst the objects *tum-tum-tum-tum-tum-tum*,
   (2b) *so-and-so* indeed possesses the property *blah-blah-blah-blah-blah-blah*, and
   (0c) *so-and-so* indeed fails to possess the property *bleh-bleh-bleh-bleh-bleh-bleh.*

3. From the point of view of pure logic, what we are doing, in giving a dis-proof by providing a counter-example against the statement $(\star)$, is to **prove the negation** $(\sim\star)$ **of the statement** $(\star)$, which reads:—

   $(\sim\star)$ '**There exists some** *so-and-so* **amongst the objects** *tum-tum-tum-tum-tum-tum* **such that** *so-and-so possesses the property*

   $$\textit{blah-blah-blah-blah-blah-blah-blah-blah-blah}$$

   and *so-and-so does* **not** *possess the property*

   $$\textit{bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh}$$

   ,

4. This is an illustration of the ideas above with a daily life example.

   Imagine the statement $(\star)$ is made:—

   $(\star)$ *Let x be a CUHK student. Suppose x has taken MATH1010. Then x has taken MATH1030.*

   To dis-prove the statement $(\star)$, all we have to do is to unveil a CUHK student who has taken MATH1010 and who has not taken MATH1030, and then verify by checking the student's transcript.

   The most difficult work is in fact the preparatory step: to make an educated guess on where to look for such a student. (A good starting point is to look at the study schemes to see whether there is a programme for which MATH1010 is a major requirement but MATH1030 is not even a major elective.)

5. These are illustrations of the ideas above from school maths.

   (a) We want to dis-prove the statement

   $(\star_a)$ *Let x be a real number. Suppose $x^2 > 0$. Then $x^3 > 0$.*

   A dis-proof by counter-example against $(\star_a)$ is given here:—

   - Take $x = -1$. Note that x is a real number.
     Note that $x^2 = 1$. Then $x^2 > 0$.
     Note that $x^3 = -1$. Then $x^3 \leq 0$. Therefore '$x^3 > 0$' is not true.

   (b) We want to dis-prove the statement

   $(\star_b)$ *Let x, y be real numbers. Suppose $x^3 - x = y^3 - y$. Then $x = y$.*

   A dis-proof by counter-example against $(\star_b)$ is given here:—

   - Take $x = 1, y = -1$. Note that x, y are real number.
     Note that $x^3 - x = 0$, and $y^3 - y = 0$. Then $x^3 - x = y^3 - y$.
     Also note that $x \neq y$. Then '$x = y$' is not true.