1.4.1 Appendix: Mathematical induction.

0. The material in this appendix is supplementary.

1. In this course you will sometimes encounter arguments that are obtained through an application of the method of **mathematical induction**.

   Such a method may be considered when we want to prove a statement, say, ($\star$), that can be expressed in the form

   ($\star$) (*Suppose blih-blih-blih. Then*) **for any integer** n **no less than (some fixed integer)** N,

   $$\underbrace{\textit{bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh-bloh}}_{\textit{a passage about the symbol } n \textit{ which will become a statement on its own once a concrete integer is `substituted' into } n} .$$

   Such a passage *bloh-bloh-bloh-...-bloh-bloh-bloh* about the symbol n which will become a statement on its own once an integer is 'substituted' into n is called a **predicate with variable** n.

   Sometimes the predicate *bloh-bloh-bloh-...-bloh-bloh-bloh* takes the form of a conditional, which reads:—

   - **if** $\underbrace{\textit{blah-blah-blah-blah-blah-blah-blah-blah-blah}}_{\textit{a passage about the symbol } n}$ **then** $\underbrace{\textit{bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh}}_{\textit{a passage about the symbol } n}$.

   For convenience of discussion, we borrow the idea of 'functional notations' to provide a label for such a predicate *bloh-bloh-bloh-...-bloh-bloh-bloh*. For instance, we may label it as, say, $P(n)$, and then as a consequence, the statement ($\star$) can be labelled as

   ($\star$) (*Suppose blih-blih-blih. Then*) **for any integer** n **no less than (some fixed integer)** N, *the predicate* $P(n)$ *holds.*

2. It must be noted that the method of mathematical induction is not applicable for the proof of every mathematical statement.

   For a statement that cannot be expressed in the form ($\star$), mathematical induction is not applicable.

   Below are some statements from school maths on which the method of mathematical induction is applicable:—

   (a) *For any positive integer* n, $\underbrace{\textit{the equality } 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)}{4} \textit{ holds}}_{\textit{predicate } P(n) \textit{ with variable } n}$.

   (b) *Suppose* α *is a complex number not equal to* 1. *Then for any positive integer* n,

   $$\underbrace{\textit{the equality } 1 + \alpha + \alpha^2 + \cdots + \alpha^n = \frac{1-\alpha^{n+1}}{1-\alpha} \textit{ holds}}_{\textit{predicate } P(n) \textit{ with variable } n}.$$

   (c) *For any positive integer* n, $\underbrace{\textit{the inequality } 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n} \textit{ holds.}}_{\textit{predicate } P(n) \textit{ with variable } n}$

   (d) *For any integer* n *no less than* 2,

   $$\underbrace{\textit{if } a_1, a_2, \cdots, a_n \textit{ are positive integers then } 2^{a_1+a_2+\cdots+a_n} = 2^{a_1}2^{a_2}\cdots 2^{a_n}}_{\textit{predicate } P(n) \textit{ with variable } n}.$$

   In this course, many results that we come across can be put in the same form.

3. When we apply the method of mathematical induction to prove a certain statement, what we mean is that, under the 'standing assumption' in the passage of argument, we are going to do as described below:—

   (1) We declare that we can re-express what we are going to immediately verify in the form

   ($\star$) '**For any integer** n **no less than (some fixed integer)** N, *the predicate* $P(n)$ *holds*',

   This declaration is usually given through the introduction of the predicate $P(n)$ with variable n.

   (2) We verify the statement $P(N)$.

   (3) We verify the statement that reads:—

'*For any integer k no less than N, if the statement $P(k)$ is true, then the statement $P(k+1)$ is true.*'

(4) We declare (in view of what has been done in (2) and (3)) that according to the Principle of Mathematical Induction, $P(m)$ is true for any integer m no less than N.

We illustrate this process with an application of the method of mathematical induction to prove this statement:—

Suppose α is a complex number not equal to 1. Then for any positive integer n,

$$\text{the equality } 1+\alpha+\alpha^2+\cdots+\alpha^n = \frac{1-\alpha^{n+1}}{1-\alpha} \text{ holds.}$$

Argument.

Suppose α is a complex number not equal to 1.

- Denote by $P(n)$ the predicate
$$1+\alpha+\alpha^2+\cdots+\alpha^n = \frac{1-\alpha^{n+1}}{1-\alpha}.$$
- [We verify $P(1)$.]
Note that
$$1+\alpha^1 = 1+\alpha = \frac{1-\alpha^2}{1-\alpha} = \frac{1-\alpha^{1+1}}{1-\alpha}.$$
Hence $P(1)$ is true.
- [We verify the statement '*for any positive integer k, if the statement $P(k)$ is true then the statement $P(k+1)$ is true*'.]
Pick any positive integer k. Suppose $P(k)$ is true. [We now ask whether $P(k+1)$ will be true as a consequence of $P(k)$.]
Note that
$$\begin{aligned}(1+\alpha+\alpha^2+\cdots+\alpha^k+\alpha^{k+1}) - \frac{1-\alpha^{k+2}}{1-\alpha} &= (1+\alpha+\alpha^2+\cdots+\alpha^k)+\alpha^{k+1} - \frac{1-\alpha^{k+2}}{1-\alpha} \\ &= \frac{1-\alpha^{k+1}}{1-\alpha}+\alpha^{k+1}-\frac{1-\alpha^{k+2}}{1-\alpha} \quad (\text{by } P(k)) \\ &= \frac{(1-\alpha^{k+1})+\alpha^{k+1}(1-\alpha)-(1-\alpha^{k+2})}{1-\alpha} \\ &= \frac{1-\alpha^{k+1}+\alpha^{k+1}-\alpha^{k+2}-1+\alpha^{k+2}}{1-\alpha} \\ &= 0\end{aligned}$$
Then $(1+\alpha+\alpha^2+\cdots+\alpha^k+\alpha^{k+1}) = \dfrac{1-\alpha^{k+2}}{1-\alpha}$.
Hence $P(k+1)$ is true.
- According to the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n.

4. The method of mathematical induction is 'driven by' the Principle of Mathematical Induction, which is a mathematical statement whose validity is left unquestioned. (There are a few such statements in mathematics; mathematicians call them **axioms**.)

Principle of Mathematical Induction.

Let $P(n)$ be a predicate with variable n.

Let N be an integer. Suppose both conditions (†), (‡) are satisfied:—

(†) *$P(N)$ is true.*

(‡) *For any integer $k \geq N$, if $P(k)$ is true then $P(k+1)$ is true.*

Then $P(m)$ is true for each integer $m \geq N$.