1.3.3 Exercise: Transpose, symmetry and skew-symmetry.

1. Let $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ p & u & x \\ q & v & y \\ r & w & z \end{bmatrix}$, in which $p, q, r, u, v, w, x, y, z$ are some numbers.

   (a) i. Write down A^t.

   ii. What is the first row of $2A^t + 3B$?

   (b) It is known that $AB = I_3$.
   Determine the values of u, v, w.

2. Suppose A is an $(m \times n)$-matrix, B, C are $(n \times p)$-matrices, and D is a $(p \times q)$-matrix.

   By directly using the definition, or using results that have been introduced/proved already, prove the statements below:—

   (a) $[A(B + C)]^t = B^t A^t + C^t A^t$.

   (b) $(ABD)^t = D^t B^t A^t$.

3. Let A be a square matrix.

   (a) Prove that $(A^2)^t = (A^t)^2$.

   (b) Apply mathematical induction to prove that $(A^n)^t = (A^t)^n$ for any positive integer n.

4. Let A_1, A_2 be matrices with m rows.

   With reference to the definition, verify that $[\ A_1 \mid A_2\ ]^t = \left[\begin{array}{c} {A_1}^t \\ \hline {A_2}^t \end{array}\right]$.

   Remark. More generally:—

   - whenever $A_1, A_2, \cdots, A_p$ are matrices with the same number of rows, $[\ A_1 \mid A_2 \mid \cdots \mid A_p\ ]^t = \left[\begin{array}{c} {A_1}^t \\ \hline {A_2}^t \\ \hline \vdots \\ \hline {A_p}^t \end{array}\right]$, and
   - whenever $B_1, B_2, \cdots, B_p$ are matrices with the same number of columns, $\left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_p \end{array}\right]^t = [\ {B_1}^t \mid {B_2}^t \mid \cdots \mid {B_p}^t\ ]$.

   We will sometimes use the special cases of these results in which the A_j's are column vectors, and the $B_k' s$ are row vectors.

5. Let $A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$.

   For each k, denote the k-th row of A by $\mathbf{a}_k$.

   Note that $A^t = [\ {\mathbf{a}_1}^t \mid {\mathbf{a}_2}^t \mid {\mathbf{a}_3}^t \mid {\mathbf{a}_4}^t \mid {\mathbf{a}_5}^t \mid {\mathbf{a}_6}^t \mid {\mathbf{a}_7}^t\ ]$ by definition of transpose.

   (a) What are the respective values of $\mathbf{a}_1{\mathbf{a}_1}^t$, $\mathbf{a}_2{\mathbf{a}_2}^t$, ..., $\mathbf{a}_7{\mathbf{a}_7}^t$?

   (b) Suppose $i \neq j$. Take for granted this observation:—
   There is exactly one k for which the k-th entries of $\mathbf{a}_i, \mathbf{a}_j$ are both 1.
   What is the value of $\mathbf{a}_i{\mathbf{a}_j}^t$?

   (c) Hence show that $AA^t = \alpha I_7 + \beta J_7$ for some appropriate numbers α, β whose value you have to name explicitly.
   Here J_7 stands for the (7×7)-square matrix whose entries are all 1.

Remark. Can you adapt what is done above to $B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$?

6. Let a, b, c, d be real numbers, and A be the (2×2)-square matrix given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   Suppose $AA^t = A^tA$ and $a \neq d$.

   Prove that A is symmetric.

7. Prove the statement below:—

   The zero $(n \times n)$-square matrix is the only $(n \times n)$-square matrix which is both symmetric and skew-symmetric.

   Remark. When it is formally formulated, this statement reads:—

   Let A be an $(n \times n)$-square matrix. Suppose A is both symmetric and skew-symmetric. Then $A = \mathcal{O}_{n \times n}$.

8. Let A, B be symmetric square matrices.

   (a) Prove the statement (♯):—

   (♯) *Suppose $AB = BA$. Then AB is symmetric.*

   (b) Is the converse of (♯) true? Justify your answer.

9. Let A be a $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij} for each i, j.

   Suppose A is skew-symmetric.

   With direct reference to definition, show that $a_{kk} = 0$ for each $k = 1, 2, \cdots, n$.

10. Prove the statement below:—

    Suppose A is a skew-symmetric $(n \times n)$-square matrix, and C is an $(n \times p)$-matrix. Then C^tAC is a skew-symmetric $(p \times p)$-square matrix.

11. Let $\mathbf{u}, \mathbf{v}$ be column vectors with n entries, and A be the $(n \times n)$-square matrix given by $A = \mathbf{u}\mathbf{v}^t - \mathbf{v}\mathbf{u}^t$.

    (a) Show that A is skew-symmetric.

    (b) Suppose $\mathbf{u}^t\mathbf{u} = \mathbf{v}^t\mathbf{v} = 1$ and $\mathbf{u}^t\mathbf{v} = p$.

    i. Show that $A^2 = \alpha(\mathbf{u}\mathbf{v}^t + \mathbf{v}\mathbf{u}^t) + \beta(\mathbf{u}\mathbf{u}^t + \mathbf{v}\mathbf{v}^t)$, in which α, β are some numbers whose respective values may depend on that of p. You have to name the values of α, β explicitly.

    ii. Show that $A^3 = \gamma A$, in which β is some number whose value may depend on that of p. You have to name the value of γ explicitly.

    Remark. We are identifying a (1×1)-matrix with the number which is the only entry of that (1×1)-matrix.

12. Let A, B be $(n \times n)$-square matrices. Suppose A is symmetric and B is skew-symmetric.

    For each of the matrices below, determine if it is symmetric or skew-symmetric or neither. Justify your answer with an appropriate argument.

    (a) $AB + BA$
    (b) $AB - BA$
    (c) A^2
    (d) B^2
    (e) $A^2B^4A^2$
    (f) $A^3B^4A^3$
    (g) $A^4B^5A^4$
    (h) $A^3B^5A^3$

13. (a) Let A be an $(m \times n)$-matrix with real entries. Suppose that $\mathbf{x}^tA\mathbf{y} = 0$ for any column vectors $\mathbf{x}$ with m real entries, and any column vectors $\mathbf{y}$ with n real entries. Prove that $A = \mathcal{O}$.

    (b) Let B be an $(n \times n)$-square matrix with real entries.

    Suppose B is symmetric, and suppose $\mathbf{u}^tB\mathbf{u} = 0$ for any column vector $\mathbf{u}$ with n real entries.

    Prove that $B = \mathcal{O}$.

    Hint. Find a way to make use of the result in the previous part.

    (c) Give a counter-example to shown that the statement below is false, and justify your answer:—

    Suppose C is an $(n \times n)$-square matrix with real entries. Suppose $\mathbf{u}^tC\mathbf{u} = 0$ for any column vector $\mathbf{u}$ with n entries. Then $C = \mathcal{O}$.