1.3.2 Appendix: Words and phrases indicating logical content of statements.

0. The material in this appendix is supplementary.

1. Some words (or pairs of words) and phrases appear very often in mathematical statements:

   - 'not',
   - 'and',
   - 'or',
   - 'if ... then',
   - 'suppose ... then',
   - 'assume ... then',
   - 'if and only if',
   - 'these statements are logically equivalent'
   - 'let',
   - 'for any' (or 'for all', or 'for each', or 'for every')
   - 'each' (or 'every', or 'all', or 'any'),
   - 'for some',
   - 'there exist some', (or 'there is some' or 'there is at least one'),
   - 'there is at most one',
   - 'there is some unique'.

   These words/phrases explain the content of a statement by indicating the 'logical relations' amongst various components in statements.

   They must not be ignored (and certainly not confused with each other) when you read a statement; otherwise you will end up mis-understanding of the content of the statement.

   Here we explain the meaning of those of them which appear very often in this course, and how they are used in general.

2. **The words 'not', 'and', 'or'.**

   In mathematics, the words '*not*', '*and*' are understood in the same way and used in the same way as they are in daily language. Care must be taken with the word '*or*'.

   Greater care must be taken when the words '*not*', '*and*', '*or*' appear close-by in the same passage.

   (a) A passage which reads
      - *blah-blah-blah-blah-blah-blah* **and** *bleh-bleh-bleh-bleh-bleh-bleh*

      informs the reader of a **simultaneity**:—
      - '*blah-blah-blah-blah-blah-blah*' happens to be true.
      - '*bleh-bleh-bleh-bleh-bleh-bleh*' also happens to be true, **simultaneously**.

   (b) A passage which reads
      - *blah-blah-blah-blah-blah-blah* **or** *bleh-bleh-bleh-bleh-bleh-bleh*

      informs the reader of three **possibilities**:—
      - '*blah-blah-blah-blah-blah-blah*' happens to be true, but '*bleh-bleh-bleh-bleh-bleh-bleh*' does not happen to be true.
      - '*bleh-bleh-bleh-bleh-bleh-bleh*' happens to be true, but '*blah-blah-blah-blah-blah-blah*' does not happen to be true.
      - '*blah-blah-blah-blah-blah-blah*' and '*bleh-bleh-bleh-bleh-bleh-bleh*' happen to be true, simultaneously.

   (c) **De Morgan's Laws.**

      i. A passage which reads
         - *It is* **not** *true that* (*blah-blah-blah* **or** *bleh-bleh-bleh*).

         means the same thing as
         - (*blah-blah-blah is* **not** *true*) **and** (*bleh-bleh-bleh is* **not** *true*).

      ii. A passage which reads
         - *It is* **not** *true that* (*blah-blah-blah* **and** *bleh-bleh-bleh*).

         means the same thing as
         - (*blah-blah-blah is* **not** *true*) **or** (*bleh-bleh-bleh is* **not** *true*).

      These two 'rules' are collectively known as **De Morgan's Laws**.

   (d) **Distributive Laws.**

      i. A passage which reads
         - (*blah-blah-blah* **or** *bleh-bleh-bleh*) **and** *blih-blih-blih*

         means the same thing as

- (*blah-blah-blah* **and** *blih-blih-blih*) **or** (*bleh-bleh-bleh* **and** *blih-blih-blih*)

ii. A passage which reads

- (*blah-blah-blah* **and** *bleh-bleh-bleh*) **or** *blih-blih-blih*

means the same thing as

- (*blah-blah-blah* **or** *blih-blih-blih*) **and** (*bleh-bleh-bleh* **or** *blih-blih-blih*)

These two 'rules' are collectively known as **Distributive Laws**. In this course we do not need to use them very often.

(e) A 'daily life' illustration of the above is given here.

Imagine that we know how the students Alan and Ben perform in MATH1030:—

- *Alan gets A.* (So Alan does not get B.)
- *Ben gets B.* (So Ben does not get A.)

i. These statements are true:—
   - *Alan gets A* **and** *Ben gets B.*
   - *Alan gets A* **or** *Ben gets B.*
   - *Alan gets A* **or** *Ben gets A.*

ii. These statements are false:—
   - *Alan gets A* **and** *Ben gets A.*
   - *Alan gets B* **or** *Ben gets A.*

iii. We can form a statement that reads:—
   - '*it is* **not** *true that* (*Alan gets B* **or** *Ben gets A*).'

   According to De Morgan Laws', this statement means the same thing as:—
   - '(*Alan does* **not** *get B*), **and** (*Ben does* **not** *get A*).'

3. **'If ... then ...', converse, and contrapositive**

We have encountered many times a statement (especially in the various theorems) that reads:—

($\star$) **Suppose** *blah-blah-blah.* **Then** *bleh-bleh-bleh.*

Or:—

($\star'$) **If** *blah-blah-blah*, **then** *bleh-bleh-bleh.*

Such a statement is called a **conditional** with **assumption** '*blah-blah-blah*' and **conclusion** '*bleh-bleh-bleh*'.

(a) Interchanging the position occupied by '*blah-blah-blah*', '*bleh-bleh-bleh*' in ($\star$), we form the **converse** of ($\star$):

($\star\star$) **Suppose** *bleh-bleh-bleh.* **Then** *blah-blah-blah.*

As we stated earlier, the converse of ($\star$) is not the same as the statement ($\star$), in the sense that whether the converse of ($\star$) is true or not has nothing to do with whether ($\star$) itself is true or false.

(b) Simultaneously interchanging the position occupied by '*blah-blah-blah*', '*bleh-bleh-bleh*' in ($\star$) and inserting the word '*not*' into both '*blah-blah-blah*', '*bleh-bleh-bleh*', we form the **contrapositive** of ($\star$):

($\star''$) **Suppose** *it is* **not** *true that bleh-bleh-bleh.* **Then** *it is* **not** *true that blah-blah-blah.*

The contrapositive ($\star''$) of the statement ($\star$) is the same as the statement ($\star$).

(c) The classic illustration of the above is the one below in Euclidean geometry:—

- **Pythagoras' Theorem.**
  Suppose $\angle C$ is Triangle ABC is a right angle. Then $AC^2 + BC^2 = AB^2$.
- **Converse of Pythagoras' Theorem.**
  Suppose $AC^2 + BC^2 = AB^2$. Then $\angle C$ is Triangle ABC is a right angle.
- **Contrapositive of Pythagoras' Theorem.**
  Suppose $AC^2 + BC^2 \neq AB^2$. Then $\angle C$ is Triangle ABC is not a right angle.
- **Contrapositive of the converse of Pythagoras' Theorem.**
  Suppose $\angle C$ is Triangle ABC is not a right angle. Then $AC^2 + BC^2 \neq AB^2$.

4. **'If and only if', 'logical equivalence', and 'dictionaries' between concepts.**

(a) We can form a new statement, labelled ($\star\star\star$) below, by 'joining' ($\star$), ($\star\star$) together with '*and*':—

($\star\star\star$) (**If** *blah-blah-blah* **then** *bleh-bleh-bleh*) **and** (**if** *bleh-bleh-bleh* **then** *blah-blah-blah*).

The statement $(\star\star\star)$ is often presented as:—

$(\star\star\star')$ *blah-blah-blah* **if and only if** *bleh-bleh-bleh.*

Or as:—

$(\star\star\star'')$ *The statements below are* **logically equivalent**:—

(†) *blah-blah-blah.*

(††) *bleh-bleh-bleh.*

A practical (and mathematically correct) way to interpret the content of the statement $(\star\star\star)$ is:

- '*blah-blah-blah*', '*bleh-bleh-bleh*' are **simultaneously true** , or they are **simultaneously false**.

This is however not the friendlies interpretation to use, when you are trying to give an argument for $(\star\star\star)$.

(b) The safest way to give an argument for $(\star\star\star)$ is to write down a pair of separate arguments, one for

- '**if** *blah-blah-blah* **then** *bleh-bleh-bleh*',

and the other for

- '**if** *bleh-bleh-bleh* **then** *blah-blah-blah*'.

(c) Though the presentation of $(\star\star\star)$ in the form of $(\star\star\star'')$ looks rather clumsy, it is in imitation of $(\star\star\star'')$ that can present the logical equivalence of several statements, such as:—

$(\star\cdots\star)$ *The statements below are* **logically equivalent**:—

(†) *blah-blah-blah.*

(††) *bleh-bleh-bleh.*

(†††) *blih-blih-blih.*

⋮

(††···†) *bloh-bloh-bloh.*

By such a statement we mean:

- '*blah-blah-blah*', '*bleh-bleh-bleh*', '*blih-blih-blih*', ..., '*bloh-bloh-bloh*' are **simultaneously true** , or they are **simultaneously false**.

(d) One of the key features in this course is to build up 'dictionaries' between two (or many) concepts/tools.

Through these 'dictionaries', we know that the same definition/result/method can be presented using seemingly unrelated concepts/tools.

When we are faced with difficulties dealing a problem with of one set of concepts/tools alone, we may to translate the problem into an 'equivalent' problem through an appropriate 'dictionary'. It could happen that we immediately knew how to solve the 'equivalent' problem.

These 'dictionaries' are usually presented in the form of a result whose conclusion reads as $(\star\cdots\star)$.

5. **'Let', 'for any', 'each', ...**

We have encountered statements that read:—

- **For any** *so-and-so* **amongst the objects** *tum-tum-tum,* **if** *so-and-so possesses the property blah-blah-blah* **then** *so-and-so possesses the property bleh-bleh-bleh.*
- **Each** *so-and-so* **amongst the objects** *tum-tum-tum* **which possesses the property** *blah-blah-blah* **will (also) possess the property** *bleh-bleh-bleh.*

For all practical purposes, they can be regarded as a very compact presentation of the statement below:—

- **Let** *so-and-so* **be amongst the objects** *tum-tum-tum.* **Suppose** *so-and-so possesses the property blah-blah-blah.* **Then** *so-and-so possesses the property bleh-bleh-bleh.*

The very compact presentation is often used when we formulate defining conditions in a definition.

6. **'There exist some', 'there is some', 'there is at least one', 'for some', ...**

In this course we have already encountered a few statements of the form (and will encounter more of these on the way):

- **There is some** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** ...
- **There exist some** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** ...

- **There is at least one** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** *...*
- *So-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** *...* **for some** *so-and-so* **amongst the objects** *tum-tum-tum.*

They mean the same thing. The chain of words '*there is some*' is to be understood in its literal sense.

(a) In school mathematics, you rarely encounter such a 'formulation', because you could usually say more in such a situation, pinpointing explicitly what 'exists' by naming a 'concrete object' with the relevant properties.

In this course, we need this kind of 'formulation' because very often it may be un-necessary or undesirable (or both) to pinpoint exactly what exists by naming a 'concrete object' with the relevant properties. Confirming that such an object exists can be good enough in many a theoretical discussion.

(b) Illustration from school maths:—

In school maths, we know these statement below are true:—

- *The equation '$x^2 - 3x + 2 = 0$' with unknown x has a solution in the reals, such as* 1.
- *Any pair of positive real numbers a, b give rise to a geometric progression with* 5 *terms starting at a and ending at b, such as a, $\sqrt[4]{a^3b}$, $\sqrt{ab}$, $\sqrt[4]{ab^3}$, b.*

We can present the respective statements with the same content as:—

- **There is some** *real number x, say, $x = 1$,* **such that** *the equality $x^2 - 3x + 2 = 0$.*
- *Given any pair of positive real numbers a, b,* **there exists some** *sequence with* 5 *terms, namely, a, $\sqrt[4]{a^3b}$, $\sqrt{ab}$, $\sqrt[4]{ab^3}$, b,* **such that** *the sequence is a geometric progression starting at a and ending at b.*

In these examples, the 'concrete objects' named are 'the real number 1', 'the sequence a, $\sqrt[4]{a^3b}$, $\sqrt{ab}$, $\sqrt[4]{ab^3}$, b' respectively.

But if we are content in knowing that such objects exist without naming them explicitly in the respective case, it will be perfectly fine to write:—

- **There is some** *real number x* **such that** *the equality $x^2 - 3x + 2 = 0$.*
- *Given any pair of positive real numbers a, b,* **there exists some** *sequence with* 5 *terms* **such that** *the sequence is a geometric progression starting at a and ending at b.*

(c) Freeing ourselves the task of naming things explicitly allows us more flexibility.

For instance, it will make perfect sense for us to make claims like:—

- **There exists some** *real number x* **such that** $x^6 + 2x^2 + 3x + 4 = 0$.
- **There exists some** *real-valued function f of one real variable* **such that** $f'(x) = \exp(x^2)$ *for any real number x.*

7. **'There is at most one'.**

Sometimes we encounter a statement of the form

(‡) **There is at most one** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** *...*

The statement (‡) must not be confused with:—

- **There is some** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** *...*

The words '*there is at most one* in the statement (‡) are to be understood in its literal sense.

The statement (‡) means the same thing as:

(‡′) **Let** *so-and-so, such-and-such be* **amongst the objects** *tum-tum-tum.*

Suppose that *each of so-and-so, such-and-such* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** *...* .

Then *so-and-so* **is equal to** *such-and-such.*

To prove a statement like (‡), it is in fact better to read it as (‡′).

8. **'There exists some unique', 'there is exactly one'.**

In this course we sometimes encounter a statement of the form:—

(*EU*) **There exists some unique** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** *...*

Or:—

- **There is exactly one** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** ...

Such a statement (EU) is known as an **existence-and-uniqueness statement**, and it is obtained when we join the two unrelated statements $(E), (U)$ with '**and**':—

(E) **There exists some** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** ...

(U) **There is at most one** *so-and-so* **amongst the objects** *tum-tum-tum* **such that** *so-and-so* **possesses the property** *blah-blah-blah* **and possesses the property** *bleh-bleh-bleh* **and** ...

We will refer to $(E), (U)$ respectively as the **existence part**, **uniqueness part** of the statement (EU).