### 1.3.1 Appendix: Miscellanies on theoretical mathematics.

0. The material in this appendix is supplementary.

1. In this course you are exposed to the style and the format modern theoretical mathematics is presented.

   Although many concepts and results covered in *linear algebra* were discovered more than a century ago and, like what you are learning in *calculus*, were related to very concrete mathematical problems, the subject gained prominence only after the mid-20th century.

   For this reason, the style and the format of presentation are heavily influenced by the development in various fields in *pure mathematics*, and of *mathematical language* in the first half of the 20th century, and would look quite remote from what you were used to in *school mathematics*.

   It will take some effort for you to get used to the type of content you are going to encounter in this course, especially if your experience in mathematics back in school days was mostly about 'methods, recipes, procedures for doing calculations' instead of 'reasoning and argument'.

   Up to this point you should have seen a few passages labelled 'definitions', 'theorems', and 'proofs'. You should have realized that there are some definite formats in these passages, which help arrange the content, and present the 'logical relation' between the various parts of the content.

   We describe these formats below. Being consciously aware of the formats will help you understand the material more efficiently and more effectively.

2. **What are 'theorems' and 'lemmas'? What is their (usual) format (in this course)?**

   We have encountered many a passage of mathematics which describes a mathematical result (and labelled 'theorem' or 'lemma'), and which is presented in this **format**:—

   ($\star$) '(**Let/Suppose** *blah-blah-blah.*) **Suppose** *blah-blah-blah.* (Also **suppose** *blah-blah-blah.* Further **suppose** *blah-blah-bhah.*)

   **Then** *bleh-bleh-bleh.* (Also, *bleh-bleh-bleh.* Moreover, *bleh-bleh-bleh.*)'

   Such a passage is called a **(mathematical) statement**.

   The content in the portion(s) '*blah-blah-blah*' is known as the **assumption** in the statement.

   The content in the portion(s) '*bleh-bleh-bleh*' is known as the **conclusion** in the statement.

   The first thing you should do when you encounter a statement like ($\star$) is to figure out what its assumption is, and what its conclusion is.

3. **Mathematical statements.**

   Though you might not be familiar with the phrase '(mathematical) statement', you have had much working experience with statements since you began to encounter mathematics.

   This is probably the first serious mathematical statement that you encountered:

   $$1 + 1 = 2.$$

   Later on you also encountered statements like:—

   - *Let $ABC$ be a triangle. Suppose $\angle C$ is right. Then $AC^2 + BC^2 = AB^2$.*
   - *Let $ABC$ be a triangle. Suppose $AC^2 + BC^2 = AB^2$. Then $\angle C$ is right.*

   You would have been told these two statements were Pythagoras' Theorem and its converse.

   Or perhaps this as well:—

   - *Let $r$ be a real number. Suppose $-1 < r < 1$. Then $1 + r + r^2 + r^3 + \cdots = \dfrac{1}{1-r}$.*

   Mathematical statements are 'basic blocks' in a mathematical discussion, when people communicate with each other on ideas in mathematics.
   In general, a mathematical statement is true, or false, but never 'both true and false'.

   Most of the time we are interested in true mathematical statements.
   ('$1 + 2 = 4$' is perfectly fine as a mathematical statement. However, because it is false, it is not interesting.)

   True (and content-wise relevant and important) mathematical statements in an area are usually referred to as 'theorems'. For example, in Euclidean geometry, because of the importance of what are called Pythagoras' Theorem and its converse, they are named as such.

4. **Various styles of presentation for theorems.**

   Different mathematicians may adopt different styles when they present the same statement, say, $(\star)$, with its assumption and its conclusion, although these styles are variant of the above. Below are some examples of various ways of presenting $(\star)$:—

   - **If** $\underbrace{\textit{blah-blah-blah} \text{ and } \textit{blah-blah-blah} \text{ and } \ldots}_{\text{all the blah-blah-blah's}}$ **then** $\underbrace{\textit{bleh-bleh-bleh} \text{ and } \textit{bleh-bleh-bleh} \text{ and } \ldots}_{\text{all the bleh-bleh-bleh's}}$.
   - **For any** *blah-blah-blah*, **if** $\underbrace{\textit{blah-blah-blah} \text{ and } \ldots}_{\text{all the blah-blah-blah's not already stated}}$ **then** $\underbrace{\textit{bleh-bleh-bleh} \text{ and } \textit{bleh-bleh-bleh} \text{ and } \ldots}_{\text{all the bleh-bleh-bleh's}}$.
   - **For any** *blah-blah-blah* such that $\underbrace{\textit{blah-blah-blah} \text{ and } \ldots}_{\text{all the blah-blah-blah's not already stated}}$,

     **then** $\underbrace{\textit{bleh-bleh-bleh} \text{ and } \textit{bleh-bleh-bleh} \text{ and } \ldots}_{\text{all the bleh-bleh-bleh's}}$.
   - Given that $\underbrace{\textit{blah-blah-blah} \text{ and } \textit{blah-blah-blah} \text{ and } \ldots}_{\text{all the blah-blah-blah's}}$, it will **follow** that $\underbrace{\textit{bleh-bleh-bleh} \text{ and } \textit{bleh-bleh-bleh} \text{ and } \ldots}_{\text{all the bleh-bleh-bleh's}}$.
   - $\underbrace{\textit{blah-blah-blah} \text{ and } \textit{blah-blah-blah} \text{ and } \ldots}_{\text{all the blah-blah-blah's}} \Longrightarrow \underbrace{\textit{bleh-bleh-bleh} \text{ and } \textit{bleh-bleh-bleh} \text{ and } \ldots}_{\text{all the bleh-bleh-bleh's}}$.

5. **How to understand a 'theorem'?**

   In plain words, the statement $(\star)$ is saying:—

   - When '*blah-blah-blah-blah-blah-blah*' happens to be true, it will as a consequence, happen that '*bleh-bleh-bleh-bleh-bleh-bleh*' is true as well.

   The statement $(\star)$ as a whole is saying **none of these**:—

   - '*blah-blah-blah*' is true.
   - '*bleh-bleh-bleh*' is true.
   - '*blah-blah-blah*' and '*bleh-bleh-bleh*' are both true.
   - When '*bleh-bleh-bleh*' is true, it will as a consequence, happen that '*blah-blah-blah*' is true as well.
   - When '*blah-blah-blah*' is not true, it will as a consequence, happen that '*bleh-bleh-bleh*' is not true as well.

   (a) A 'daily life' example which illustrates the point above can be obtained when you take:—
   - *blah-blah-blah* to be '*I am attending an MATH1030 lecture*',
   - *bleh-bleh-bleh* to be '*I am inside the CUHK campus*'.

   (b) An elementary mathematical example which illustrates the same point can be obtained when you take:—
   - *blah-blah-blah* to be '*Triangle ABC is equilateral*',
   - *bleh-bleh-bleh* to be '*Triangle ABC is isosceles*'.

6. **What do people do when they want to convince others a certain statements is true?**

   When we are confident that such a statement like $(\star)$ is true, we will try to '**prove**' (or '**verify**') it. And once the statement $(\star)$ is 'proved', we call it a **theorem** (or sometimes a **lemma**).

   Such a process is for convincing anyone, friend or foe alike, that what we believe to be true, is indeed true. This is (usually) accomplished by the presentation of an appropriate chain of mathematical statements which are previously proved results (and thus are safely taken for granted to be true) and which are logically 'linked together'.

   Depending on the content of the theorem to be proved, such a chain of mathematical statements might be short, but could also be very lengthy.

   In the latter situation, people tend to first identify the key statements in such a chain. If such a key statement is itself important/useful enough to deserve being highlighted, it is stated elsewhere (usually earlier) and proved on its own as lemmas.

7. **Direct proofs.**

   The most basic type of proofs for a statement like $(\star)$ is known as **direct proof**, which takes this **format**:—

   $\underbrace{\text{Suppose } \textit{blah-blah-blah-blah-blah-blah.}}_{\text{the whole assumption in the statement } (\star).}$

   Material that may be eligible and relevant to the deducing of (part of the) conclusion, to be filled in. It may include:—

- part or all of the assumption in $(\star)$,
- content of previously proved results,
- content of definition of objects involved in the assumption in $(\star)$,
- calculations of various sorts.

$\underbrace{\text{Hence it follows that } \textit{bleh-bleh-bleh-bleh-bleh-bleh.}}_{\text{(part of the) conclusion that has been deduced from the above}}$

More material that may be eligible and relevant to the deducing of (part of the) conclusion, to be filled in. It may further include what the part of the conclusion that has been deduced.

$\underbrace{\text{Hence it follows that } \textit{bleh-bleh-bleh-bleh-bleh-bleh.}}_{\text{(part of the) conclusion that has been deduced from the above}}$

So forth and so on, until the whole conclusion in $(\star)$ is deduced.

You will find that most of the proofs appearing in this course are 'direct proofs' (though some may be short and easy, and some may be lengthy and difficult).

There are proofs which are not 'direct proofs' but which are equally valid. They are mainly: the **proof-by-contradiction argument**, and the **contra-positive argument**.

Whenever it is possible, we will give 'direct proofs' when we have to prove something in this course.

And if you want to give a proof for something, you are encouraged to present 'direct proofs'.

Very often the statements under question involve equalities amongst matrices and vectors. When you take the approach of proof-by-contradiction argument or the contra-positive argument, you will lose the advantage of working with equalities. Moreover, you will very often have to carry the word 'not' from one line of argument to another; this will make an attempted argument more prone to logical mistakes.

8. **Dis-proving something which is (believed to be) false.**

   When we are confident that a statement like $(\star)$ is false, we will try to 'dis-prove' it by providing a **counter-example** against the statement concerned. (We will encounter this kind of argument later.)

9. **Definitions.**

   We have also encountered many a passage of mathematics which we call a **definition**.

   The purpose of a definition is to propose a name (that we will adhere to in all future discussion) for a specific type of objects or a special collection of properties that can be attributed to various objects (that will appear very often in all future discussion).

   It does not make sense to say things like '*this definition is true/false*'. We do not 'prove' a definition.

   A definition is judged on whether it is 'good and useful'. A 'good and useful' definition is one:—

   - which is rich because there are plenty of examples (and non-examples), and
   - which indeed helps simplify theoretical discussions.

   You had encountered many definitions in school maths. Below are some examples of definitions that you must have encountered:—

   - *A triangle is a plane figure made up of three non-collinear points and the three line segments joining the three respective pairs of points.*
   - *A sphere is a closed surface in space whose points are all at the same distance from some common point in space.*
   - *Two plane figures are said to be congruent to each other exactly when they are of the same shape and of the same size.*
   - *Two distinct lines on the plane are said to be parallel to each other exactly when, no matter how much they are extended, they will not meet each other.*

   Most likely you would have glossed over them in your school days (as they seemed to play little role in helping you do 'calculations').

   If this is the case, you have to consciously change your approach now.

   Without knowing (and understanding) definitions, you will not be in any position to understand the content of the theorems which are built on the definitions involved, and will not be able to know what you are trying to calculate with reference to the respective theorems.

   Very often new definitions are built in terms of previously introduced definitions. To understand the content of new definitions, you need to first understanding the previously introduced definitions.

10. **Two standard formats of definitions.**

Many a definition will take either **format** below:—

(a) Format 1 (very often used for naming an object).

(**Suppose** $\underbrace{\textit{blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih}}_{\text{condition under which the object to be defined may be discussed.}}$.

**Then**) a $\underbrace{\textit{so-and-so-and-so}}_{\text{name of the object to be defined}}$ is a

$$\underbrace{\textit{blah-blah-blah-blah-blah-blah-blah-blah-blah-blah-blah-blah-blah-blah}}_{\text{concrete object with understood properties, giving the meaning of the phrase `so-and-so-and-so'}}$$

which satisfies (*or* which is given by)

$$\underbrace{\textit{bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh}}_{\text{further elaboration (if necessary) on the meaning of the phrase `so-and-so-and-so'}}.$$

(b) Format 2, very often used for naming a collection of properties.

**Suppose**

$$\underbrace{\textit{blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih}}_{\text{objects for which, and conditions under which, the collection of properties to be defined may be discussed.}}.$$

**Then** a

$$\underbrace{\textit{blah-blah-blah}}_{\text{object which may possess the properties collectively known as `so-and-so-and-so'}}$$

is **said to be**

$$\underbrace{\textit{so-and-so}}_{\text{name assigned to the collection of properties}}$$

**if and only if**

$$\underbrace{\textit{bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh}}_{\text{meaning of `blah-blah-blah is so-and-so', in terms of what is already understood}}.$$

The content in '*bleh-bleh-bleh-...-bleh-bleh-bleh*' is known as the **defining condition** (or **defining criterion**) in the definition for *blah-blah-blah* being *so-and-so*.

When '*bleh-bleh-bleh-...-bleh-bleh-bleh*' is very lengthy, we may present the same definition as:—

**Suppose**

$$\underbrace{\textit{blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih-blih}}_{\text{objects for which, and conditions under which, the collection of properties to be defined may be discussed.}}.$$

**Then** a

$$\underbrace{\textit{blah-blah-blah}}_{\text{object which may possess the properties collectively known as `so-and-so-and-so'}}$$

is **said to be**

$$\underbrace{\textit{so-and-so}}_{\text{name assigned to the collection of properties}}$$

**if and only if** the statement (♯) below hold:—

$$(\sharp): \quad \underbrace{\textit{bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh-bleh}}_{\text{meaning of `blah-blah-blah is so-and-so', in terms of what is already understood}}.$$