1.3 Transpose, symmetry and skew-symmetry.

0. *Assumed background.*

   - 1.1 *Matrices, matrix addition, and scalar multiplication for matrices.*
   - 1.2 *Matrix multiplication.*

   Abstract. We introduce:—

   - the notion of transpose,
   - the notions of symmetry and skew-symmetry.

   In the *appendices*, we digress onto the notion of definition, theorem, proof, and the format which dictates how they are to be read, and in the meaning of words and phrases which indicate the logical content of statements.

1. **Definition. (Transpose of a matrix.)**

   Let A be an $(m \times n)$-matrix, whose (i, j)-th entry is denoted by a_{ij}.

   The **transpose of** A *is the $(n \times m)$-matrix whose (k, ℓ)-th entry is given by $a_{\ell k}$.*

   It is denoted by A^t.

   Remark. In symbolic terms, what this definition says is:—

   $$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \text{ then } A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

2. **Example (1). (Transpose of a matrix.)**

   Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}$.

   Then $A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B^t = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$ and $C^t = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

   (a) Note that $A + B = \begin{bmatrix} 2 & 5 & 3 \\ 2 & 2 & 3 \end{bmatrix}$. Then $(A + B)^t = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$.

   We have $A^t + B^t = \cdots = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$. So $(A + B)^t = A^t + B^t$ (in this example).

   (b) Note that $AC = \cdots = \begin{bmatrix} 4 & 13 \\ 2 & 7 \end{bmatrix}$. Then $(AC)^t = \begin{bmatrix} 4 & 2 \\ 13 & 7 \end{bmatrix}$.

   We have $C^tA^t = \cdots = \begin{bmatrix} 4 & 2 \\ 13 & 7 \end{bmatrix}$. So $(AC)^t = C^tA^t$ (in this example).

3. **Theorem (1). (Basic properties of transpose.)**

   The statements below hold:—

   (1) *Suppose A is an $(m \times n)$-matrix. Then $(A^t)^t = A$.*

   (2) *Suppose A, B are $(m \times n)$-matrices. Then $(A + B)^t = A^t + B^t$.*

   (3) *Suppose A is an $(m \times n)$-matrix, and λ is a number. Then $(\lambda A)^t = \lambda A^t$.*

   (4) *Suppose A is an $(m \times n)$-matrices, and C is an $(n \times p)$-matrix. Then $(AC)^t = C^tA^t$.*

   Proof of Statement (4) of Theorem (1).

   Suppose A is a $(m \times n)$-matrix, and C is an $(n \times p)$-matrix. (So AC is an $(m \times p)$-matrix, and $(AC)^t$ is a $(p \times m)$-matrix.)

   (By definition, A^t is an $(n \times m)$-matrix, and C^t is a $(p \times n)$-matrix. So C^tA^t is well-defined as a $(p \times m)$-matrix.)

   Denote the (i, j)-th entry of A by a_{ij}. Denote the (k, ℓ)-th entry of C by $c_{k\ell}$.

   Fix any $\ell = 1, 2, \cdots, p$ and $i = 1, 2, \cdots, m$.

- By the definition of matrix multiplication, the (i, ℓ)-th entry of AC is given by $\sum_{j=1}^{n} a_{ij}c_{j\ell}$.

  Then, by the definition of transpose, the (ℓ, i)-th entry of $(AC)^t$ is given by $\sum_{j=1}^{n} a_{ij}c_{j\ell}$.

- By the definition of transpose, for each $j = 1, 2, \cdots, n$, the (ℓ, j)-th entry of C^t is $c_{j\ell}$, and the (j, i)-th entry of A^t is a_{ij}.

  Then, by the definition of matrix multiplication, the (ℓ, i)-th entry of C^tA^t is given by $\sum_{j=1}^{n} a_{ij}c_{j\ell}$.

Hence $(AC)^t = C^tA^t$.

Proof of Statements (1), (2), (3) of Theorem (1). Exercise. (Imitate what is done above.)

4. **Definition. (Symmetric matrix and skew-symmetric matrix.)**

   Suppose A is an $(n \times n)$-square matrix. Then:—

   (1) *A is said to be* **symmetric** *if and only if $A^t = A$.*

   (2) *A is said to be* **skew-symmetric** *if and only if $A^t = -A$.*

5. **Example (2). (Examples and non-examples on symmetric matrices and skew-symmetric matrices.)**

   (a) The $(n \times n)$-zero matrix is a symmetric matrix. It is also a skew-symmetric matrix.

   (b) The identity matrix is a symmetric matrix. It is not skew-symmetric.

   (c) Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 6 \end{bmatrix}$.

   Note that $A^t = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 6 \end{bmatrix} = A$. Then A is symmetric.

   Note that $A^t \neq -A$. Then A is not skew-symmetric.

   (d) Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$.

   Note that $A^t = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -A$. Then A is skew-symmetric.

   Note that $A^t \neq A$. Then A is not symmetric.

   (e) Let $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

   Note that $B^t = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

   We have $B^t \neq B$. Then B is not symmetric.
   We have $B^t \neq -B$. Then B is not skew-symmetric.

   (f) Let $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   Note that $B^t = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   We have $B^t \neq B$. Then B is not symmetric.
   We have $B^t \neq -B$. Then B is not skew-symmetric.

6. **Lemma (2).**

   Suppose A is a square matrix. Then:—

   (1) *$A + A^t$ is symmetric.*

   (2) *$A - A^t$ is skew-symmetric.*

   Proof of Lemma (2).

   Suppose A is a square matrix.

(1) We have $(A+A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$.
Then, by definition of symmetric matrix, $A + A^t$ is symmetric.

(2) We have $(A-A^t)^t = [A+(-A^t)]^t = A^t + (-A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t)$.
Then, by definition of skew-symmetric matrix, $A - A^t$ is skew-symmetric.

7. **Theorem (3).**

Suppose A is a square matrix. Then there are some unique square matrices B, C such that B is symmetric, C is skew-symmetric, and $A = B + C$.

Proof of Theorem (3).

Suppose A is a square matrix.

[We have two tasks, which are (α), (β) below:—

(α) Conceive some appropriate symmetric matrix, and some appropriate skew-symmetric matrix, respectively labelled B, C in the subsequent consideration, which we hope will satisfy $A = B + C$.

(β) Then we verify for such a pair of matrices B, C two things:—

(1) The equality '$A = B + C$' holds indeed.

(2) If some symmetric matrix P and some skew-symmetric matrix Q also satisfy $A = P + Q$, then $P = B$ and $Q = C$.

We proceed with (α), and follow up with (β).

But how to proceed with (α)?]

[*Roughwork.*

According to Lemma (2), we have a pair of symmetric matrix and skew-symmetric matrix determined by A alone:—

- $A + A^t$ is a symmetric matrix.
- $A - A^t$ is a skew-symmetric matrix.

However, because $(A + A^t) + (A - A^t) = 2A$, they are not the respective B, C that we hope for. But we are getting close.]

Define $B = \frac{1}{2}(A + A^t)$, and $C = \frac{1}{2}(A - A^t)$.

Note that B is symmetric, and C is skew-symmetric. (Why? Apply Lemma (2) and Theorem (1).)

- We have $B + C = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) = A$.
- Suppose P is a symmetric matrix, Q is a skew-symmetric matrix, and $A = P + Q$.
  [Ask: Is it true that $B = P$ and $C = Q$?]
  By assumption, $P^t = P$ and $Q^t = -Q$. Then $A^t = (P+Q)^t = P^t + Q^t = P - Q$.
  Now we have $2P = (P+Q) + (P-Q) = A + A^t$. Then $P = \frac{1}{2}(A + A^t) = B$.
  We also have $2Q = (P+Q) - (P-Q) = A - A^t$. Then $Q = \frac{1}{2}(A - A^t) = C$.