1.2.1 Exercise: Matrix multiplication.

In some of the questions below, you will need the respective notions of *idempotency* and *involutoricy*. Their respective definitions are given below:—

Let C be a square matrix.

(1) *We say that C is* **idempotent** *if and only if $C^2 = C$.*

(2) *We say that C is* **involutoric** *if and only if C^2 is the identity matrix.*

1. Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$.

   Compute AB and BA.

2. Let $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$.

   Compute $(AB)C$ and $A(BC)$.

3. Let $A = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 2 & 8 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} a & 2 \\ 1 & b \\ b & a \\ -1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & c \\ 4 & d \end{bmatrix}$.

   Suppose $AB = C$. Find the values of a, b, c, d.

4. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ -5 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & -1 & -7 \\ -2 & 1 & 3 & 4 \\ 3 & 2 & 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 6 & 0 & -6 \\ -1 & 2 & 4 & 5 \\ 4 & 3 & 2 & 3 \end{bmatrix}$.

   (a) Is it true that $B = C$? Justify your answer.

   (b) Compute AB and AC.

   (c) Is it true that $AB = AC$? Justify your answer.

5. Let $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 4 \\ 1 & -2 & -4 \\ -1 & 2 & 4 \end{bmatrix}$.

   (a) Compute AB and BA.

   (b) Verify that A, B are idempotent.

   (c) Let p, q be real numbers.

      i. Verify that $(pA + qB)^2 = p^2A + q^2B$.

      ii. Apply mathematical induction to prove that $(pA + qB)^n = p^nA + q^nB$ for any positive integer n.

6. Let r be a number, and A be the matrix given by $A = \begin{bmatrix} 3-2r & 2-2r & 2-2r \\ -1+r & r & -1+r \\ -3+r & -3+r & -2+r \end{bmatrix}$.

   Verify that A is involutary.

7. Let w, x, y, z be numbers.

   Show that

$$\begin{bmatrix} x-y & y-z & z-w & w-x \\ w-x & x-y & y-z & z-w \end{bmatrix} - \begin{bmatrix} x-w & y-x & z-y & w-z \\ y-x & z-y & w-z & x-w \end{bmatrix} = \begin{bmatrix} 1 \\ p \end{bmatrix} \begin{bmatrix} q & r & -q & -r \end{bmatrix},$$

   in which p, q, r are some appropriate numbers whose values may be dependent on the values of w, x, y, z. You have to name p, q, r explicitly.

8. Let A, B be square matrices of the same size. Name some appropriate numbers a, b, c, d for which the equality

$$(2A + 3B)(A - B) = aA^2 + bAB + cBA + dB^2$$

   holds.

9. Let A be the (3×3)-square matrix given by

$$A = \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix}.$$

Verify that $(aI_3 - A)(bI_3 - A)(cI_3 - A) = \mathcal{O}_{3\times 3}$.

10. (a) Compute the matrix product $\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

(b) Hence express in matrix notation the equations (with unknown x, y) below. Give your answer in the form 'blah-blah-blah $= 0$'.

i. $x^2 + 9xy + y^2 + 8x + 5y + 2 = 0$.

ii. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

iii. $xy = 25$.

iv. $y^2 = 12x$.

11. Let a be a number, and $A = \begin{bmatrix} 0 & a & a^2 & a^3 \\ 0 & 0 & a & a^2 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Compute A^2, A^3, A^4.

12. (a) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Find AB.

(b) Let A be an $(m \times n)$-matrix. Suppose every entry of A is 1.

Let B be an $(n \times n)$-square matrix given by

$$B = \begin{bmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n & 1 & \vdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{bmatrix}.$$

Show that $AB = \alpha A$, in which α is a number whose value might depend on that of n. You have to determine the explicit value of α.

13. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$.

For each statement below, determine whether it is true or false. Justify your answer.

(a) $(A + B)^2 = A^2 + 2AB + B^2$.

(b) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

14. Prove the statement below:—

Let A, B be square matrices of the same size. Suppose $AB = A$ and $BA = B$. Then each of A, B is idempotent.

15. (a) Prove the statement ($\sharp$):—

($\sharp$) *Let A be an $(n \times n)$-square matrix. Suppose A is involutoric. Then $(I_n + A)(I_n - A) = \mathcal{O}_{n\times n}$.*

(b) Is the converse of ($\sharp$) true or not? Justify your answer.

16. Let A be an $(n \times n)$-square matrix. Suppose $A^2 = \mathcal{O}_{n\times n}$.

Apply mathematical induction to prove the statement below:—

For any positive integer m, the matrix equality $A(I_n + A)^m = A$ holds.

17. Let A be an $(m \times n)$-matrix and B be an $(n \times p)$-matrix.

    Suppose the k-th and ℓ-th rows of A are identical.

    Show that the k-th and ℓ-th rows of AB are identical.

    Hint. It may be easy to think of matrix multiplication (and organize your argument) in terms of appropriate blocks of rows/columns.

18. In this question, A stands for an $(m \times n)$-matrix, B stands for an $(n \times p)$-matrix, and C stands for an $(m \times p)$-matrix.

    Furthermore, $\mathbf{x}$ stands for the column vector resultant from adding the columns of B together, and $\mathbf{y}$ stands for the column vector resultant from adding the columns of C together.

    (a) Prove the statement ($\sharp$):—

    ($\sharp$) *Suppose $AB = C$. Then $A\mathbf{x} = \mathbf{y}$.*

    Hint. Can you relate B and $\mathbf{x}$ through an appropriate matrix equality involving matrix multiplication? How about C and $\mathbf{y}$?

    (b) Is the statement ($\flat$) true?

    ($\flat$) *Suppose $A\mathbf{x} = \mathbf{y}$. Then $AB = C$.*

    Give an appropriate justification for your answer, by providing a proof for the statement ($\flat$), or providing a counter-example against ($\flat$).

19. Prove the statement below (by directly applying the definition of matrix equality or otherwise):—

    Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are column vectors each with m entries, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are numbers.

    Then $\left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n\ \right] \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots \lambda_n\mathbf{v}_n.$

    Remark. This is a useful special case of a general result:—

    Suppose $A_1, A_2, \cdots, A_q$ are matrices all with m rows, and with $n_1, n_2, \cdots, n_q$ columns respectively.

    Further suppose $B_1, B_2, \cdots, B_q$ are matrices all with p columns, and with $n_1, n_2, \cdots, n_q$ rows respectively.

    Then $\left[\ A_1 \mid A_2 \mid \cdots \mid A_q\ \right] \begin{bmatrix} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_q \end{bmatrix} = A_1B_1 + A_2B_2 + \cdots + A_qB_q.$

20. By directly applying the definition of matrix multiplication, or otherwise, prove the statements below:—

    Suppose A is an $(m \times n)$-matrix.

    Further suppose $B = \left[\ B_1 \mid B_2 \mid \cdots \mid B_s\ \right]$, in which $B_1, B_2, \cdots, B_s$ are matrices all with n rows and with $p_1, p_2, \cdots, p_s$ columns respectively.

    Then $AB = \left[\ AB_1 \mid AB_2 \mid \cdots \mid AB_s\ \right]$.

    Remark. Below is an analogous result:—

    Suppose $A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_s \end{bmatrix}$, *in which $A_1, A_2, \cdots, A_s$ be matrices all with n columns and with $m_1, m_2, \cdots, m_s$ rows respectively.*

    Further suppose B is an $(n \times p)$-matrix.

    Then $AB = \begin{bmatrix} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_sB \end{bmatrix}.$

21. Let

    $$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad Q = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \qquad R = \begin{bmatrix} aa' & bb' \\ cc' & dd' \end{bmatrix},$$

    in which $a, b, c, d, a', b', c', d'$ are some numbers.

    Suppose the equality $PQ = R$ holds.

(a) Show that

$$\begin{cases} bc' &= 0 \\ ab' + bd' &= bb' \\ ca' + dc' &= cc' \\ cb' &= 0 \end{cases}$$

Remark. The result described in this question tells you that it is highly non-trivial for the product of two (2×2)-square matrices to be equal to the (2×2)-square matrix obtained by just multiplying the corresponding (i, j)-th entries together for each i, j.

(b) Now further suppose $P = Q$. Show that

$$b = c = 0 \text{ or } (b = 0 \text{ and } a + d = c) \text{ or } (c = 0 \text{ and } a + d = b).$$