1.2 Matrix multiplication.

0. *Assumed background.*

   - 1.1 *Matrices, matrix addition, and scalar multiplication for matrices.*

   Abstract. We introduce:—

   - matrix multiplication (first for row vectors to column vectors from the left, then in the general situation through presentation in blocks),
   - properties of matrix multiplication,
   - notions of square matrix and identity matrix,
   - notion of positive powers of square matrices,
   - presentation of matrix multiplication in terms of blocks.

1. **Definition. (Multiplication of row vector to column vector from the left.)**

   Let A be a row vector with n entries, and B be a column vector with n entries.

   Suppose $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$, *and* $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

   Then we define the product AB to be the (1×1)-matrix

   $$\begin{bmatrix} a_1b_1 + a_2b_2 + \cdots + a_nb_n \end{bmatrix}.$$

   For future convenience we abuse notations to confuse as the number $a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

2. **Definition. (Multiplication of matrix to column vector from the left.)**

   Let A be an $(m \times n)$-matrix, and B be a column vector with n entries.

   Suppose $A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix}$, *in which A_k stands for the k-th row of A for each k.*

   We define the product AB to be the column vector with m entries, given by

   $$AB = \begin{bmatrix} A_1B \\ A_2B \\ \vdots \\ A_mB \end{bmatrix}.$$

   (For each k, the k-th entry of AB is the number A_kB.)

   Remark. Denote the (i, j)-th entry of A by a_{ij}.

   Denote the j-th entry of B by b_j.

   Then the k-th entry of AB is given by the number $a_{k1}b_1 + a_{k2}b_2 + \cdots + a_{kn}b_n$.

   Writing out the entries in the matrices explicitly, we have

   $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11}b_1 & + & a_{12}b_2 & + & \cdots & + & a_{1n}b_n \\ a_{21}b_1 & + & a_{22}b_2 & + & \cdots & + & a_{2n}b_n \\ \vdots & & \vdots & & & & \vdots \\ a_{m1}b_1 & + & a_{m2}b_2 & + & \cdots & + & a_{mn}b_n \end{bmatrix}.$$

3. **Definition. (Matrix multiplication.)**

   Let A be an $(m \times n)$-matrix, and B be an $(n \times p)$-matrix.

   Suppose $B = \begin{bmatrix} B_1 \mid B_2 \mid \cdots \mid B_p \end{bmatrix}$, *in which B_ℓ is the ℓ-th column of B for each ℓ.*

   We define the product AB to be the $(m \times p)$-matrix given by

   $$AB = \begin{bmatrix} AB_1 \mid AB_2 \mid \cdots \mid AB_p \end{bmatrix}.$$

(For each ℓ, the ℓ-th column of AB is the column vector AB_ℓ with m entries.)

Remark. Denote the (i, j)-th entry of A by a_{ij}. Denote the (k, ℓ)-th entry of B by $b_{k\ell}$.

Denote the i-th row of A by A_i.

Then the (i, ℓ)-th entry of AB is given by the number $A_i B_\ell = a_{i1}b_{1\ell} + a_{i2}b_{2\ell} + \cdots + a_{in}b_{n\ell}$, and

$$AB = \left[\begin{array}{c|c|c|c} A_1B_1 & A_1B_2 & \cdots & A_1B_p \\ A_2B_1 & A_2B_2 & \cdots & A_2B_p \\ \vdots & \vdots & & \vdots \\ A_mB_1 & A_mB_2 & \cdots & A_mB_p \end{array}\right] = \begin{bmatrix} \sum_{j=1}^n a_{1j}b_{j1} & \sum_{j=1}^n a_{1j}b_{j2} & \cdots & \sum_{j=1}^n a_{1j}b_{jp} \\ \sum_{j=1}^n a_{2j}b_{j1} & \sum_{j=1}^n a_{2j}b_{j2} & \cdots & \sum_{j=1}^n a_{2j}b_{jp} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^n a_{mj}b_{jp} & \sum_{j=1}^n a_{mj}b_{j2} & \cdots & \sum_{j=1}^n a_{mj}b_{jp} \end{bmatrix} = \left[\begin{array}{c} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{array}\right].$$

4. **Example (1). (Matrix multiplication.)**

(a) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}$.

Write $A_1 = [\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ ]$, $A_2 = [\ 2\ \ 3\ \ 4\ \ 5\ \ 6\ ]$, $A_3 = [\ 3\ \ 4\ \ 5\ \ 6\ \ 7\ ]$, $A_4 = [\ 4\ \ 5\ \ 6\ \ 7\ \ 8\ ]$.

Write $B_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $B_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$.

We have $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline A_3 \\ \hline A_4 \end{array}\right]$, $B = [\ B_1 \mid B_2\ ]$.

Then

$$AB = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline A_3 \\ \hline A_4 \end{array}\right] [\ B_1 \mid B_2\ ] = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \\ A_3B_1 & A_3B_2 \\ A_4B_1 & A_4B_2 \end{bmatrix} = \begin{bmatrix} 40 & 115 \\ 50 & 150 \\ 60 & 185 \\ 70 & 220 \end{bmatrix}.$$

(b) Let $A = \begin{bmatrix} 1 & -1 & 1 & 6 & 1 \\ 6 & 4 & 1 & 4 & -2 \\ 2 & 3 & 2 & -1 & 3 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -4 & 1 \\ -1 & 1 & 2 \\ 4 & 2 & -3 \\ 6 & 3 & 4 \end{bmatrix}$.

We have

$$AB = \begin{bmatrix} 1 & -1 & 1 & 6 & 1 \\ 6 & 4 & 1 & 4 & -2 \\ 2 & 3 & 2 & -1 & 3 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 2 & -4 & 1 \\ -1 & 1 & 2 \\ 4 & 2 & -3 \\ 6 & 3 & 4 \end{bmatrix} = \cdots = \begin{bmatrix} 28 & 20 & -18 \\ 17 & -13 & -44 \\ 20 & -3 & 12 \\ 10 & -1 & -3 \end{bmatrix}.$$

5. **Theorem (1). (Distributive Laws for addition and multiplication of matrices.)**

(1) *Suppose A is an $(m \times n)$-matrix and B, C are $(n \times p)$-matrices. Then $A(B + C) = (AB) + (AC)$.*

(2) *Suppose A, B are $(m \times n)$-matrices and C is an $(n \times p)$-matrix. Then $(A + B)C = (AC) + (BC)$.*

Remark on notations. We will dispense with the brackets in '$(AB) + (AC)$', '$(AC) + (BC)$', and simply write '$AB + AC$', '$AC + BC$' respectively.

Proof of Statement (1) of Theorem (1).

Suppose A is an $(m \times n)$-matrix and B, C are $(n \times p)$-matrices.

(a) Suppose $m = 1$ and $p = 1$ for the moment. (So A is a row vector and B, C are column vectors.)

For each j, denote the $(1, j)$-th entry of A by a_{1j}.

For each k, denote the $(k, 1)$-th entries of B, C respectively by b_{k1}, c_{k1}.

The $(k, 1)$-th entry of B, C is given by $b_{k1} + c_{k1}$.

By definition,

$$\begin{array}{rcl} AB & = & [\ a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1n}b_{n1}\ ] \\ AC & = & [\ a_{11}c_{11}+a_{12}c_{21}+\cdots+a_{1n}c_{n1}\ ] \\ A(B+C) & = & [\ a_{11}(b_{11}+c_{11})+a_{12}(b_{21}+c_{21})+\cdots+a_{1n}(b_{n1}+c_{n1})\ ] \end{array}$$

Then

$$\begin{array}{rcl} A(B+C) & = & [\ a_{11}(b_{11}+c_{11})+a_{12}(b_{21}+c_{21})+\cdots+a_{1n}(b_{n1}+c_{n1})\ ] \\ & = & [\ (a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1n}b_{n1})+(a_{11}c_{11}+a_{12}c_{21}+\cdots+a_{1n}c_{n1})\ ] \\ & = & [\ a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1n}b_{n1}\ ]+[\ a_{11}c_{11}+a_{12}c_{21}+\cdots+a_{1n}c_{n1}\ ] \\ & = & AB+AC \end{array}$$

(b) We now still suppose $p=1$, but leave the value of m un-restricted.
For each i, denote the i-th row of A by A_i.
By the calculation above, we have $A_i(B+C)=A_iB+A_iC$.
Then

$$A(B+C)=\begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix}(B+C)=\begin{bmatrix} A_1(B+C) \\ A_2(B+C) \\ \vdots \\ A_m(B+C) \end{bmatrix}=\begin{bmatrix} A_1B+A_1C \\ A_2B+A_2C \\ \vdots \\ A_mB+A_mC \end{bmatrix}=\begin{bmatrix} A_1B \\ A_2B \\ \vdots \\ A_mB \end{bmatrix}+\begin{bmatrix} A_1C \\ A_2C \\ \vdots \\ A_mC \end{bmatrix}=AB+AC.$$

(c) We now leave the values of m,p un-restricted.
For each j, denote the j-th columns of B,C by B_j,C_j respectively.
By the calculation above, we have $A(B_j+C_j)=AB_j+AC_j$.
Then

$$\begin{array}{rcl} A(B+C) & = & A\ [\ B_1+C_1 \mid B_2+C_2 \mid \cdots \mid B_p+C_p\ ] \\ & = & [\ A(B_1+C_1) \mid A(B_2+C_2) \mid \cdots \mid A(B_p+C_p)\ ] \\ & = & [\ AB_1+AC_1 \mid AB_2+AC_2 \mid \cdots \mid AB_p+AC_p\ ] \\ & = & [\ AB_1 \mid AB_2 \mid \cdots \mid AB_p\ ]+[\ AC_1 \mid AC_2 \mid \cdots \mid AC_p\ ]=AB+AC \end{array}$$

Proof of Statement (2) of Theorem (1). This is left as an exercise. (Imitate what is done above.)

6. **Theorem (2).**

   Suppose A is an $(m\times n)$-matrix and B is an $(n\times p)$-matrix. Suppose λ is a number.

   Then $\lambda(AB)=(\lambda A)B=A(\lambda B)$.

   Remark on notations. We will dispense with the brackets in '$\lambda(AB)$', '$(\lambda A)B$', and simply write 'λAB'.

   Proof of Theorem (2). Exercise.

7. **Definition. (Square matrix.)**

   A matrix with the same number of rows and columns is called a **square matrix***.*

8. **Theorem (3). ('Existence and uniqueness' of 'multiplicative identity' for matrix multiplication.)**

   There is a unique $(n\times n)$-square matrix M such that for any $(n\times n)$-square matrix A, the equalities '$MA=A$', '$AM=A$' hold.

   Proof of Theorem (3).

   Define $M=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$.

   Define $\delta_{ij}=\begin{cases} 1 & \text{if} \quad i=j \\ 0 & \text{if} \quad i\neq j \end{cases}$. ('$\delta_{ij}$' is called the Kronecker symbol.)

   Then the (i,j)-th entry of M is δ_{ij} for each i,j.

   [We intend to verify two things:

(1) *The equalities '$MA = A$', '$AM = A$' hold for any $(n \times n)$-matrix A.*

(2) *If some $(n \times n)$-matrix P possesses the property that both equalities '$PA = A$', '$AP = A$' hold for any $(n \times n)$-matrix A', then $P = M$.*

We proceed with (1), (2) separately.]

(1) Let A be an $(n \times n)$-matrix with its (i, j)-th entry given by a_{ij}.

  * By the definition of matrix multiplication, the (i, j)-th entry of MA is given by

$$\delta_{i1}a_{1j} + \delta_{i2}a_{2j} + \delta_{i3}a_{3j} + \cdots + \delta_{in}a_{nj} = \delta_{ii}a_{ij} = a_{ij}.$$

  Hence $MA = A$.

  * By the definition of matrix multiplication, the (i, j)-th entry of AM is given by

$$a_{i1}\delta_{1j} + a_{i2}\delta_{2j} + a_{i3}\delta_{3j} + \cdots + a_{in}\delta_{nj} = a_{ij}\delta_{jj} = a_{ij}.$$

  Hence $AM = A$.

(2) Let P be an $(n \times n)$-matrix. Suppose $AP = A = PA$ for any $(n \times n)$-matrix A.
Then in particular, because M is an $(n \times n)$-matrix, we have $MP = M = PM$.
Since P is an $(n \times n)$-matrix, we have $PM = P$ from the calculations above.
Hence $M = PM = P$.

9. **Definition. (Identity matrix.)**

For each positive integer n, the $(n \times n)$-square matrix whose (k, k)-th entry is 1 for each k and whose every other entry is 0, given explicitly by

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

is called the $(n \times n)$-identity matrix.

It is denoted by I_n.

10. **Theorem (4).**

Suppose A is an $(m \times n)$-matrix. Then the equalities '$I_m A = A$', '$AI_n = A$' hold.

Proof of Theorem (4). Exercise. (Imitate (the relevant portion of) the argument for Theorem (4).)

11. **Theorem (5). (Associativity of matrix multiplication.)**

Suppose A is an $(m \times n)$-matrix, B is an $(n \times p)$-matrix, and C is a $(p \times q)$-matrix. Then $A(BC) = (AB)C$.

Remark. We will give a proof for Theorem (5) later. (The argument is not hard. It is in the same spirit as the argument for Theorem (1), but the work in keeping track of symbols is more 'involved'.)

Because of this result, we may write '$(AB)C$', '$A(BC)$' simply as ABC, unless we want to emphasize that associativity of matrix multiplication is used.

In the light of this result, the definition below, for the notion of positive integral powers of square matrices, makes perfect sense.

12. **Definition. (Positive integral powers of square matrices.)**

Let n be a positive integer. Suppose A is a square matrix.

*The n-**th power of** A is defined to be the square matrix $\underbrace{AAA\cdots AAA}_{n \text{ copies of } A}$.*

It is denoted by A^n.

13. **Example (2). (Positive integral powers of square matrices.)**

(a) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. (Note that A itself is neither I_4 nor $-I_4$.)

We have $A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$.

(b) Let $B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. (Note that B itself is not the zero matrix.)

We have $B^2 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $B^3 = \begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $B^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $B^5 = \mathcal{O}_{5\times 5}$.

Remark. The various parts of Example (2) have told us that the statements (†), (‡) are false:—

(†) *Suppose that A is a $(n \times n)$-square matrix with real entries. Further that suppose there is some positive p such that $A^p = I_n$. Then $A = I_n$ or $A = -I_n$.*

(‡) *Suppose that B is a square matrix. Further suppose that B is not the zero matrix. Then, for each positive integer p, B^p is not the zero matrix.*

The content of part (a) of Example (2) is referred to as a counter-example against the statement (†).

The content of part (b) of Example (2) is referred to as a counter-example against the statement (‡).

14. **Proof of Theorem (5).**

Suppose A is an $(m \times n)$-matrix, B is an $(n \times p)$-matrix, and C is a $(p \times q)$-matrix.

(a) Suppose $m = 1$ and $q = 1$ for the moment. (So A is a row vector and C is a column vector.)

Suppose $A = [\ a_1 \quad a_2 \quad \cdots \quad a_n\ ]$, and $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$.

Denote the (j,k)-th entry by b_{jk}. Denote the j-th row of B by $B_{\text{row-}j}$. Denote the k-th column of B by $B_{\text{col-}k}$.

(So $B = [\ B_{\text{col-1}} \mid B_{\text{col-2}} \mid \cdots \mid B_{\text{col-}p}\ ] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} B_{\text{row-1}} \\ \hline B_{\text{row-2}} \\ \hline \vdots \\ \hline B_{\text{row-}n} \end{bmatrix}$.)

We claim that each of $(AB)C$, $A(BC)$ is the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

- We have

$$\begin{aligned} AB &= A[\ B_{\text{col-1}} \mid B_{\text{col-2}} \mid \cdots \mid B_{\text{col-}p}\ ] = [\ AB_{\text{col-1}} \mid AB_{\text{col-2}} \mid \cdots \mid AB_{\text{col-}p}\ ] \\ &= \begin{bmatrix} \sum_{j=1}^{n} a_j b_{j1} & \sum_{j=1}^{n} a_j b_{j2} & \cdots & \sum_{j=1}^{n} a_j b_{jp} \end{bmatrix} \end{aligned}$$

Then

$$(AB)C = \left(\sum_{j=1}^{n} a_j b_{j1}\right) c_1 + \left(\sum_{j=1}^{n} a_j b_{j2}\right) c_2 + \cdots + \left(\sum_{j=1}^{n} a_j b_{jp}\right) c_p = \sum_{k=1}^{p} c_k \left(\sum_{j=1}^{n} a_j b_{jk}\right) = \sum_{k=1}^{p}\sum_{j=1}^{n} a_j b_{jk} c_k.$$

So (AB)C is the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

- We have

$$BC = \begin{bmatrix} B_{\text{row-1}} \\ \hline B_{\text{row-2}} \\ \hline \vdots \\ \hline B_{\text{row-}n} \end{bmatrix} C = \begin{bmatrix} B_{\text{row-1}}C \\ \hline B_{\text{row-2}}C \\ \hline \vdots \\ \hline B_{\text{row-}n}C \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{p} b_{1k} c_k \\ \sum_{k=1}^{p} b_{2k} c_k \\ \vdots \\ \sum_{k=1}^{p} b_{nk} c_k \end{bmatrix}$$

Then

$$A(BC) = a_1 \left(\sum_{k=1}^{p} b_{1k} c_k\right) + a_2 \left(\sum_{k=1}^{p} b_{2k} c_k\right) + \cdots + a_n \left(\sum_{k=1}^{p} b_{nk} c_k\right) = \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{p} b_{jk}\right) = \sum_{j=1}^{n}\sum_{k=1}^{p} a_j b_{jk} c_k.$$

So $A(BC)$ is also the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

(b) We now leave the values of m, q un-restricted.

For each i, denote the i-th row of A by A_i. (So $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{array}\right]$.)

For each ℓ, denote the ℓ-th column of C by C_ℓ. (So $C = [\ C_1 \mid C_2 \mid \cdots \mid C_q\ ]$.)

- We have $AB = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{array}\right] B = \left[\begin{array}{c} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{array}\right]$.

  Then $(AB)C = \left[\begin{array}{c} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{array}\right] [\ C_1 \mid C_2 \mid \cdots \mid C_q\ ] = \begin{bmatrix} (A_1B)C_1 & (A_1B)C_2 & \cdots & (A_1B)C_q \\ (A_2B)C_1 & (A_2B)C_2 & \cdots & (A_2B)C_q \\ \vdots & \vdots & & \vdots \\ (A_mB)C_1 & (A_mB)C_2 & \cdots & (A_mB)C_q \end{bmatrix}$.

  The (i, ℓ)-th entry of $(AB)C$ is $(A_iB)C_\ell$ for each i, ℓ.
- We have $BC = B[\ C_1 \mid C_2 \mid \cdots \mid C_q\ ] = [\ BC_1 \mid BC_2 \mid \cdots \mid BC_q\ ]$.

  Then $A(BC) = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{array}\right] [\ BC_1 \mid BC_2 \mid \cdots \mid BC_q\ ] = \begin{bmatrix} A_1(BC_1) & A_1(BC_2) & \cdots & A_1(BC_q) \\ A_2(BC_1) & A_2(BC_2) & \cdots & A_2(BC_q) \\ \vdots & \vdots & & \vdots \\ A_m(BC_1) & A_m(BC_2) & \cdots & A_m(BC_q) \end{bmatrix}$.

  The (i, ℓ)-th entry of $A(BC)$ is $A_i(BC_\ell)$ for each i, ℓ.

By the calculations above, the equality $(A_iB)C_\ell = A_i(BC_\ell)$ holds for each i and for each ℓ.

Then $(AB)C = A(BC)$ by the definition of matrix equality.