1.1.1 Exercise: Matrices, matrix addition, and scalar multiplication for matrices.

1. (a) Write out the (6×6)-matrix A whose (i, j)-th entry a_{ij} is given by $a_{ij} = 1$ for each i, j.

   (b) Write out the (6×6)-matrix B whose (i, j)-th entry b_{ij} is given by $b_{ij} = i + j$ for each i, j.

   (c) Write out the (6×6)-matrix C whose (i, j)-th entry c_{ij} is given by $c_{ij} = \begin{cases} 1 & \text{if } i + j \text{ is even} \\ 0 & \text{otherwise} \end{cases}$.

   (d) Write out the (6×6)-matrix D whose (i, j)-th entry d_{ij} is given by $d_{ij} = (-1)^{i-j}$ for each i, j.

   (e) Write out the (6×6)-matrix E whose (i, j)-th entry e_{ij} is given by $e_{ij} = \begin{cases} -1 & \text{if } i > j \\ 0 & \text{if } i = j \\ 1 & \text{if } i < j \end{cases}$.

   (f) For the matrices A, C, D above, name some appropriate numbers p, q (if such exist) for which the equality $pC = A + qD$.

2. Let A be an (5×5)-matrix whose (i, j)-th entry is a_{ij} for each i, j.

   (a) Write out (in terms of the entries of A) the (5×5)-matrix B whose (i, j)-th entry b_{ij} is given by $b_{ij} = a_{i,6-j}$.

   (b) Write out (in terms of the entries of A) the (5×5)-matrix C whose (i, j)-th entry c_{ij} is given by $c_{ij} = a_{6-i,6-j}$.

3. Let r be a number.

   Suppose $r^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + r \begin{bmatrix} -1 & -3 \\ 1 & r-3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix} = \mathcal{O}_{2\times 2}$.

   Determine all positive values of r.

4. Suppose A, B, C are matrices of the same size.

   By directly using the definition, or using results that have been introduced/proved already, prove the statements below:—

   (a) $-(A + B) = -A - B$.

   (b) *Suppose $A + C = B + C$. Then $A = B$.*

5. Apply mathematical induction to prove the statement below:—

   Suppose A is a matrix. Then, for any positive integer n, the equality $nA = \underbrace{A + A + A + \cdots + A}_{n \text{ copies of } A}$ holds.

6. Suppose A, B are $(m \times n)$-matrices.

   Determine all possible $(m \times n)$-matrices X for which $3(X + \frac{1}{2}A) = 5(X - \frac{3}{4}B)$ holds. Leave your answer(s) in terms of A, B.

7. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

   Determine all possible (3×3)-matrices X for which the equality $X + A = 2(X - B)$ holds.