1.1 Matrices, matrix addition, and scalar multiplication for matrices.

0. *Abstract.* We introduce:—

- the notion of matrices, and the notion of equality for matrices,
- matrix addition, and its properties,
- the notions of zero matrix and additive inverse,
- scalar multiplication for matrices, and its properties,
- presentation of matrices in terms of blocks, and presentation of matrix addition and scalar multiplication in terms of blocks.

1. **Definition. (Matrices.)**

*An $(m \times n)$-**matrix** (or matrix of size m by n) with real/complex entries, with m rows and n columns, is an $(m \times n)$-rectangular array*

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{bmatrix}$$

in which the mn entries x_{ij}'s are respectively real/complex numbers.

Denote such a matrix by X. Fix any $k = 1, 2, \cdots, m$, and any $\ell = 1, 2, \cdots, n$.

(1) *The k**-th row** of the matrix X is the 'horizontal' array*

$$\begin{bmatrix} x_{k1} & x_{k2} & x_{k3} & \cdots & x_{kn} \end{bmatrix}.$$

(2) *The ℓ**-th column** of X is the 'vertical' array*

$$\begin{bmatrix} x_{1\ell} \\ x_{2\ell} \\ x_{3\ell} \\ \vdots \\ x_{m\ell} \end{bmatrix}.$$

(3) *The (k, ℓ)**-th entry** (or the (k, ℓ)**-th element**) of X is number $x_{k\ell}$.*
(It is where the k-th row and the ℓ-th column of X meet.)

Further terminologies.

A $(1 \times n)$-matrix (with just one row) is also called a **row vector** *with size n.*

An $(m \times 1)$-matrix (with just one column) is also called a **column vector** *with size m.*

2. **Example (1).**

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a (3×2)-matrix.

Its first, second and third rows are

$$\begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 5 & 6 \end{bmatrix}$$

respectively.
Its first and second columns are

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

respectively.

(b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ is a (4×4)-matrix.

Its first, second, third and fourth rows are

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 & 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 4 & 5 & 6 & 7 \end{bmatrix}$$

respectively.

Its first, second, third and fourth columns are

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$

respectively.

3. **Example (2).**

(a) Let $X = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{bmatrix}$.

The (i, j)-th entry of X is i^j.

(b) Let a be a real number, and $X = \begin{bmatrix} a & a & a \\ 0 & a & a \\ 0 & 0 & a \end{bmatrix}$.

Denote the (i, j)-th entry of X by x_{ij}.

Then $x_{ij} = \begin{cases} a & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$.

(c) Let $X = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ e & 1 & 0 & \cdots & 0 \\ e^2 & e & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^n & e^{n-1} & e^{n-2} & \cdots & 1 \end{bmatrix}$.

Denote the (i, j)-th entry of X by x_{ij}.

Then $x_{ij} = \begin{cases} e^{i-j} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$.

4. **Definition. (Equality for matrices.)**

Let A be an $(m \times n)$-matrix, with its (i, j)-th entry being a_{ij} for each i, j.

Let B be a $(p \times q)$-matrix, with its (k, ℓ)-th entry being b_{ij} for each k, ℓ.

We say that A, B are **equal (as matrices)** *if and only if:*

(1) *$m = p$ and $n = q$, and moreover,*

(2) *$a_{ij} = b_{ij}$ for all i, j.*

5. **Definition. (Addition for matrices.)**

Let A, B be $(m \times n)$-matrices with the (i, j)-th entries respectively given by a_{ij}, b_{ij} for each i, j.

We define the **sum of the matrices** *A, B to be the $(m \times n)$-matrix whose (i, j)-th entry is $a_{ij} + b_{ij}$ for each i, j.*

It is denoted by $A + B$.

*(We also read $A + B$ as the '***resultant of B added to A***'.)*

Remark. In symbols, this definition says:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

6. **Example (3). (Addition for matrices.)**

(a) $\begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} (-1) + 1 & 0 + 2 \\ 2 + (-1) & (-2) + 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 5 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1+7 & 2+5 & 3+3 \\ 4+5 & 5+3 & 6+1 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 6 \\ 9 & 8 & 7 \end{bmatrix}$.

7. **Theorem (1). (Commutativity and associativity of matrix addition.)**

(1) *Suppose A, B are $(m \times n)$-matrices. Then $A + B = B + A$.*

(2) *Suppose A, B, C are $(m \times n)$-matrices. Then $A + (B + C) = (A + B) + C$.*

Remark. By virtue of (2), we agree to write '$A + B + C$' for either '$A + (B + C)$' or '$(A + B) + C$'.

8. **Proof of Statement (1) of Theorem (1).** Suppose A, B are $(m \times n)$-matrices. Denote the respective (i, j)-th entries of A, B by a_{ij}, b_{ij} for each i, j.

   Fix any i, j.

   By the definition of matrix addition, the (i, j)-th entry of $A + B$ is $a_{ij} + b_{ij}$.

   Similarly, The (i, j)-th entry of $B + A$ is $b_{ij} + a_{ij}$.

   By the commutativity of addition for real/complex numbers, we have $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.

   Then by the definition of matrix equality, $A + B = B + A$.

   Proof of Statement (2) of Theorem (1). This is left as an exercise.

   (Imitate what is done above, using associativity of addition for real/complex numbers instead.)

9. **Theorem (2). ('Existence and uniqueness' of 'additive identity' for matrices.)**

   There is a unique $(m \times n)$-matrix Z such that for any $(m \times n)$-matrix A, the equality $A + Z = A$ holds.

   Proof of Theorem (2). Let Z be the $(m \times n)$-matrix whose entries are all 0.

   [We intend to verify two things:

   (1) *The equality $A + Z = A$ holds for any $(m \times n)$-matrix A.*

   (2) *If some $(m \times n)$-matrix Y possesses the property '$A + Y = A$ for any $(m \times n)$-matrix A', then $Y = Z$.*

   We proceed with (1), (2) separately.]

   (1) Let A be an $(m \times n)$-matrix with the (i, j)-th entry given by a_{ij} for each i, j.
   For each i, j, the (i, j)-th entry of $A + Z$ is given by $a_{ij} + 0 = a_{ij}$.
   Then by the definition of matrix addition, $A + Z = A$.

   (2) Let Y be an $(m \times n)$-matrix with the (i, j)-th entry given by y_{ij} for each i, j. Suppose $A + Y = A$ for any $(m \times n)$-matrix A.
   Then, in particular, $Z + Y = Z$.
   By the definition of matrix addition, for each i, j, we have $0 + y_{ij} = 0$. Then $y_{ij} = 0$.
   Therefore, by the definition of matrix equality, $Y = Z$.

10. **Definition. (Zero matrix.)**

    *The $(m \times n)$-matrix whose entries are all 0 is called the $(m \times n)$-**zero matrix**.*

    It is denoted by $\mathcal{O}_{m\times n}$, (or simply $\mathcal{O}$ when no confusion arises).

11. **Theorem (3). ('Existence and uniqueness' of 'additive inverse' for a matrix)**

    Suppose A is an $(m \times n)$-matrix. Then there is a unique $(m \times n)$-matrix C such that $A + C = \mathcal{O}_{m\times n}$.

    Proof of Theorem (3). Exercise, imitating the proof of Theorem (2). [We provide the beginning steps below:

    > *Suppose A is an $(m \times n)$-matrix, with its (i, j)-th entry given by a_{ij} for each i, j.*
    > *Let P be the $(m \times n)$-matrix, with its (i, j)-th entry given by $-a_{ij}$ for each i, j.*

    Now imitate the argument for Theorem (2) to verify the statements below:—

    (1) $A + P = \mathcal{O}_{m,n}$.

    (2) *If Q is an $(m \times n)$-matrix satisfying $A + Q = \mathcal{O}_{m,n}$ then $Q = P$ as matrices.*

    Fill in the detail as an exercise.]

12. **Definition. (Additive inverse, 'matrix subtraction'.)**

    Let A, B be $(m \times n)$-matrices with the (i, j)-th entries respectively given by a_{ij}, b_{ij} for each i, j.

(a) *The* **additive inverse of** A *is the* $(m \times n)$*-matrix whose* (i, j)*-th entry is given by* $-a_{ij}$ *for each* i, j.
It is denoted by $-A$.
(We also read $-A$ *as '***minus** A*'.)*

(b) *The* **difference of** B **from** A *is the* $(m \times n)$*-matrix given by the sum* $B + (-A)$.
(For each i, j*, its* (i, j)*-th entry is given by* $b_{ij} - a_{ij}$*.)*
We may write $B + (-A)$ *as* $B - A$.
(We also read $B - A$ *as the '***resultant of subtracting** A **from** B*'.)*

13. **Definition. (Scalar multiplication for matrices.)**

Let A *be* $(m \times n)$*-matrices with real/complex entries, with its* (i, j)*-th entry given by* a_{ij} *for each* i, j.

Let λ *be a real/complex number.*

The **product of the matrix** A **by the scalar** λ *is defined to be the* $(m \times n)$*-matrix whose* (i, j)*-th entry is* λa_{ij} *for each* i, j.

It is denoted by λA.

(We also read λA *as 'the* **scalar multiple of** A **by** λ*', or the '***resultant of multiplying the matrix** A **by the scalar** λ*'.)*

Remark. In symbols, this definition says:

$$\lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

14. **Example (4). (Scalar multiplication for matrices.)**

(a) $3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \\ 3 \cdot 4 & 3 \cdot 5 & 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$.

(b) $(5 \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 4 \end{bmatrix}) + (2 \begin{bmatrix} 0 & 8 \\ 7 & 0 \\ 6 & 5 \end{bmatrix}) = \begin{bmatrix} 5 & 0 \\ 0 & 10 \\ 15 & 20 \end{bmatrix} + \begin{bmatrix} 0 & 16 \\ 14 & 0 \\ 12 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 16 \\ 14 & 10 \\ 27 & 30 \end{bmatrix}$.

15. **Theorem (4). (Properties of scalar multiplication for matrices.)**

Suppose A, B *are* $(m \times n)$*-matrices, and* λ, μ *are scalars. Then:—*

(1) $\lambda(A + B) = \lambda A + \lambda B$.

(2) $(\lambda + \mu)A = \lambda A + \mu A$.

(3) $\lambda(\mu A) = (\lambda\mu)A$.

(4) $1A = A$.

(5) $(-1)A = -A$.

(6) $0A = \mathcal{O}_{m \times n}$.

Proof of Theorem (4). Exercise. (Imitate the arguments for Theorem (1).)

16. **Presentation of matrices in blocks, introduced through examples.**

Very often, for one reason or another, we like to:—

- visualize various 'rectangular blocks of entries' inside a given matrix as matrices on their own, or
- construct a matrix by putting given matrices of 'smaller sizes' alongside each other.

We introduce this idea through concrete examples.

(a) Let $A_1, A_2, \cdots, A_p$ be matrices each with m rows, and with $n_1, n_2, \cdots, n_p$ columns respectively.
The matrix $[\ A_1 \mid A_2 \mid \cdots \mid A_p\ ]$ stands for the $(m \times (n_1 + n_2 + \cdots + n_p))$-matrix whose columns from left to right are that of $A_1, A_2, \cdots, A_p$ in succession, each from left to right.
Illustration.
Let $A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}$, $A_3 = \begin{bmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{35} & a_{36} \\ a_{45} & a_{46} \end{bmatrix}$.

Then $[\ A_1 \mid A_2 \mid A_3\ ] = \left[\begin{array}{ccc|c|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{array}\right]$.

(b) Let $B_1, B_2, \cdots, B_p$ be matrices each with n columns, and with $m_1, m_2, \cdots, m_p$ rows respectively.

The matrix $\left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_p \end{array}\right]$ stands for the $((m_1 + m_2 + \cdots + m_p) \times n)$-matrix whose rows from top to bottom are that of $B_1, B_2, \cdots, B_p$ in succession, each from top to bottom.

Illustration.

Let $B_1 = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, $B_2 = \begin{bmatrix} b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$, $B_3 = \begin{bmatrix} b_{51} & b_{52} & b_{53} & b_{54} \\ b_{61} & b_{62} & b_{63} & b_{64} \end{bmatrix}$.

Then $\left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline B_3 \end{array}\right] = \left[\begin{array}{cccc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ \hline b_{41} & b_{42} & b_{43} & b_{44} \\ \hline b_{51} & b_{52} & b_{53} & b_{54} \\ b_{61} & b_{62} & b_{63} & b_{64} \end{array}\right]$.

(c) The same idea can be extended to the construction of matrices with rows and columns of blocks.

Illustration.

Let $C_{11} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix}$, $C_{12} = \begin{bmatrix} c_{14} \\ c_{24} \\ c_{34} \\ c_{44} \end{bmatrix}$, $C_{13} = \begin{bmatrix} c_{15} & c_{16} \\ c_{25} & c_{26} \\ c_{35} & c_{36} \\ c_{45} & c_{46} \end{bmatrix}$, $C_{21} = \begin{bmatrix} c_{51} & c_{52} & c_{53} \\ c_{61} & c_{62} & c_{63} \\ c_{71} & c_{72} & c_{73} \end{bmatrix}$, $C_{22} = \begin{bmatrix} c_{54} \\ c_{64} \\ c_{74} \end{bmatrix}$,

$C_{23} = \begin{bmatrix} c_{55} & c_{56} \\ c_{65} & c_{66} \\ c_{75} & c_{76} \end{bmatrix}$,

Then $\left[\begin{array}{c|c|c} C_{11} & C_{12} & C_{13} \\ \hline C_{21} & C_{22} & C_{32} \end{array}\right] = \left[\begin{array}{ccc|c|cc} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ \hline c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \\ c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} \end{array}\right]$.

17. **Theorem (5).**

Let A_{ij}, B_{ij} be matrices for each $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$.

Suppose that for each $i = 1, 2, \cdots, p$, the matrices $A_{i1}, A_{i2}, \cdots, A_{iq}, B_{i1}, B_{i2}, \cdots, B_{iq}$ have the same number of rows.

Suppose that for each $j = 1, 2, \cdots, q$, the matrices $A_{1j}, A_{2j}, \cdots, A_{qj}, B_{1j}, B_{2j}, \cdots pj$ have the same number of columns.

Define $A = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1q} \\ \hline A_{21} & A_{22} & \cdots & A_{2q} \\ \hline \vdots & \vdots & & \vdots \\ \hline A_{p1} & A_{p2} & \cdots & A_{pq} \end{array}\right]$, *and* $B = \left[\begin{array}{c|c|c|c} B_{11} & B_{12} & \cdots & B_{1q} \\ \hline B_{21} & B_{22} & \cdots & B_{2q} \\ \hline \vdots & \vdots & & \vdots \\ \hline B_{p1} & B_{p2} & \cdots & B_{pq} \end{array}\right]$.

Then $A + B = \left[\begin{array}{c|c|c|c} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1q} + B_{1q} \\ \hline A_{21} + B_{21} & A_{22} + B_{22} & \cdots & A_{2q} + B_{2q} \\ \hline \vdots & \vdots & & \vdots \\ \hline A_{p1} + B_{p1} & A_{p2} + B_{p2} & \cdots & A_{pq} + B_{pq} \end{array}\right]$.

Moreover, $\lambda A = \left[\begin{array}{c|c|c|c} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1q} \\ \hline \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2q} \\ \hline \vdots & \vdots & & \vdots \\ \hline \lambda A_{p1} & \lambda A_{p2} & \cdots & \lambda A_{pq} \end{array}\right]$

for each number λ.

Proof of Theorem (5). Omitted. (This is omitted not because it is difficult, but because it is a tedious and straightforward exercise in book-keeping.)

18. **Illustrations of the content of Theorem (5).**

(a) Let $A = \left[\begin{array}{ccc|c|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{array}\right]$, $B = \left[\begin{array}{ccc|c|cc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \end{array}\right]$.

Let $A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}$, $A_3 = \begin{bmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{35} & a_{36} \\ a_{45} & a_{46} \end{bmatrix}$.

Let $B_1 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$, $B_2 = \begin{bmatrix} b_{14} \\ b_{24} \\ b_{34} \\ b_{44} \end{bmatrix}$, $B_3 = \begin{bmatrix} b_{15} & b_{16} \\ b_{25} & b_{26} \\ b_{35} & b_{36} \\ b_{45} & b_{46} \end{bmatrix}$.

Then we have $A = [\ A_1 \mid A_2 \mid A_3\ ]$, $B = [\ B_1 \mid B_2 \mid B_3\ ]$, and

$A_1+B_1 = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \\ a_{41}+b_{41} & a_{42}+b_{42} & a_{43}+b_{43} \end{bmatrix}$, $A_2+B_2 = \begin{bmatrix} a_{14}+b_{14} \\ a_{24}+b_{24} \\ a_{34}+b_{34} \\ a_{44}+b_{44} \end{bmatrix}$, $A_3+B_3 = \begin{bmatrix} a_{15}+b_{15} & a_{16}+b_{16} \\ a_{25}+b_{25} & a_{26}+b_{26} \\ a_{35}+b_{35} & a_{36}+b_{36} \\ a_{45}+b_{45} & a_{46}+b_{46} \end{bmatrix}$.

So $A + B = [\ A_1+B_1 \mid A_2+B_2 \mid A_3+B_3\ ]$ indeed.

(b) Let $C = \left[\begin{array}{cccc} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ \hline c_{41} & c_{42} & c_{43} & c_{44} \\ \hline c_{51} & c_{52} & c_{53} & c_{54} \\ c_{61} & c_{62} & c_{63} & c_{64} \end{array}\right]$.

Let $C_1 = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$, $C_2 = [\ c_{41} \quad c_{42} \quad c_{43} \quad c_{44}\ ]$, $C_3 = \begin{bmatrix} c_{51} & c_{52} & c_{53} & c_{54} \\ c_{61} & c_{62} & c_{63} & c_{64} \end{bmatrix}$.

Then $C = \left[\begin{array}{c} C_1 \\ \hline C_2 \\ \hline C_3 \end{array}\right]$.

For each number λ, we have $\lambda C_1 = \begin{bmatrix} \lambda c_{11} & \lambda c_{12} & \lambda c_{13} & \lambda c_{14} \\ \lambda c_{21} & \lambda c_{22} & \lambda c_{23} & \lambda c_{24} \\ \lambda c_{31} & \lambda c_{32} & \lambda c_{33} & \lambda c_{34} \end{bmatrix}$, $\lambda C_2 = [\ \lambda c_{41} \quad \lambda c_{42} \quad \lambda c_{43} \quad \lambda c_{44}\ ]$, $\lambda C_3 = \begin{bmatrix} \lambda c_{51} & \lambda c_{52} & \lambda c_{53} & \lambda c_{54} \\ \lambda c_{61} & \lambda c_{62} & \lambda c_{63} & \lambda c_{64} \end{bmatrix}$.

So $\lambda C = \left[\begin{array}{c} \lambda C_1 \\ \hline \lambda C_2 \\ \hline \lambda C_3 \end{array}\right]$

(c) Let $A_{11}, A_{12}, A_{21}, A_{22}$, $B_{11}, B_{12}, B_{21}, B_{22}$ be matrices.

Suppose that:—

- the number of rows of $A_{11}, A_{12}, B_{11}, B_{12}$ are the same,
- the number of rows of $A_{21}, A_{22}, B_{21}, B_{22}$ are the same,
- the number of columns of $A_{11}, A_{21}, B_{11}, B_{21}$ are the same, and
- the number of column of $A_{12}, A_{22}, B_{12}, B_{22}$ are the same.

Define $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$.

Then $A + B = \left[\begin{array}{c|c} A_{11}+B_{11} & A_{12}+B_{12} \\ \hline A_{21}+B_{21} & A_{22}+B_{22} \end{array}\right]$.

Moreover, for each $\alpha \in \mathbb{R}$, $\alpha A = \left[\begin{array}{c|c} \alpha A_{11} & \alpha A_{12} \\ \hline \alpha A_{21} & \alpha A_{22} \end{array}\right]$.