2.4 Solving systems of linear equations.

0. *Assumed background.*

   - 2.1 *Systems of linear equations.*
   - 2.2 *Row-echelon forms and reduced row-echelon forms.*
   - 2.3 *Existence of reduced row-echelon form row-equivalent to given matrix, (and the uniqueness question).*

   Preferred to have been prepared with.

   - 1.5 *Linear combinations.*
   - 1.6 *Linear dependence and linear independence.*

   Abstract. We introduce:—

   - how to systematically determine, through the use of row-equivalence and row-echelon forms, whether a given system of linear equations is consistent or inconsistent, and whether the system has two or more solutions when it is consistent,
   - how to fully describe, through the use of reduced row-echelon forms, all solutions of a consistent system of linear equations.

1. Whenever we are given an equation (of any kind), we are interested in the three questions:—

   (1) How to determine whether the equation has any solution (with or without writing down a solution of the system explicitly), or any solution beyond the 'obvious' ones?

   (2) How to write down one solution of the the equation?

   (3) How to describe fully and systematically all solutions of the equation?

   When the equation is a system of linear equations, we can answer these questions comprehensively.

   At the heart of the answers to these questions is Theorem (1), which is a theoretical result about row-equivalent matrices.

2. **Theorem (1).**

   Let A, A' be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{b}'$ be column vectors with m entries.

   Suppose $A, \mathbf{b}$ and $A', \mathbf{b}'$ are row-equivalent under the same sequence of row operations.

   Suppose $\mathbf{t}$ is a column vector with n entries.

   Then $\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if $\mathbf{t}$ is a solution of $\mathcal{LS}(A', \mathbf{b}')$.

   Remark on terminology. In *set language*, the conclusion of Theorem (1) can be presented as:—

   - '*The solution set of $\mathcal{LS}(A, \mathbf{b})$ is equal to the solution set of $\mathcal{LS}(A', \mathbf{b}')$.*'

3. **Proof of Theorem (1).**

   Let A, A' be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{b}'$ be column vectors with m entries.

   Suppose $A, \mathbf{b}$ are respectively row-equivalent to $A', \mathbf{b}'$ under the same sequence of row operations, say,

$$A \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots\cdots\cdots \xrightarrow{\rho_k} A', \qquad \mathbf{b} \xrightarrow{\rho_1} \xrightarrow{\rho_2} \cdots\cdots\cdots \xrightarrow{\rho_k} \mathbf{b}'.$$

   - *Preparation.*
     The row operations $\rho_1, \rho_2, \cdots, \rho_k$ correspond to row-operation matrices $M[\rho_1], M[\rho_2], \cdots, M[\rho_k]$ respectively, and the equalities

$$A' = M[\rho_k] \cdots M[\rho_2]M[\rho_1]A, \qquad \mathbf{b}' = M[\rho_k] \cdots M[\rho_2]M[\rho_1]\mathbf{b}$$

     hold.
     For each ℓ, denote by $\widetilde{\rho_\ell}$ the 'reverse row operation' for ρ_ℓ.
     $\widetilde{\rho}$ corresponds to the row-operation matrix $M[\widetilde{\rho}]$, and the equality $M[\widetilde{\rho}]M[\rho] = I_p$ holds.
     Then the equalities

$$A = M[\widetilde{\rho_1}]M[\widetilde{\rho_2}] \cdots M[\widetilde{\rho_k}]A', \qquad \mathbf{b} = M[\widetilde{\rho_1}]M[\widetilde{\rho_2}] \cdots M[\widetilde{\rho_k}]\mathbf{b}'$$

     hold.

   Suppose $\mathbf{t}$ is a column vector with n entries.

(a) Suppose $\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{b})$. Then by definition, we have $A\mathbf{t} = \mathbf{b}$.
Therefore $A'\mathbf{t} = M[\rho_k] \cdots M[\rho_2] M[\rho_1] A\mathbf{t} = M[\rho_k] \cdots M[\rho_2] M[\rho_1] \mathbf{b} = \mathbf{b}'$.
Hence $\mathbf{t}$ is a solution of $\mathcal{LS}(A,\, \mathbf{b})$.

(b) Suppose $\mathbf{t}$ is a solution of $\mathcal{LS}(A',\, \mathbf{b}')$. Then $A'\mathbf{t} = \mathbf{b}'$.
We have $A\mathbf{t} = M[\widetilde{\rho_1}] M[\widetilde{\rho_2}] \cdots M[\widetilde{\rho_k}] A'\mathbf{t} = M[\widetilde{\rho_1}] M[\widetilde{\rho_2}] \cdots M[\widetilde{\rho_k}] \mathbf{b}' = \mathbf{b}$.
Hence $\mathbf{t}$ is a solution of $\mathcal{LS}(A',\, \mathbf{b}')$.

4. **Comments on the implication of Theorem (1), in the context of solving systems of linear equations.**

According to Theorem (1), to solve any given system of linear equations $\mathcal{LS}(A,\, \mathbf{b})$, it suffices for us to look for some appropriate system $\mathcal{LS}(A',\, \mathbf{b}')$ for which it happens that:—

- the respective augmented matrix representations

$$[\ A \mid \mathbf{b}\ ], \qquad [\ A' \mid \mathbf{b}'\ ]$$

  of the two systems are row-equivalent to each other, (so that $A, \mathbf{b}$ are respectively row-equivalent to $A', \mathbf{b}'$ under the same sequence of row operations),
- it is easy to read off from the system $\mathcal{LS}(A',\, \mathbf{b}')$ the answer to the question whether the system is consistent, and
- (where the system is consistent,) it is easy to read off all solutions of the system $\mathcal{LS}(A',\, \mathbf{b}')$, and to give a full and systematic description of the solutions.

Question. But what kind of $A', \mathbf{b}'$ shall we choose?

Answer. We have learnt about reduced row-echelon forms, and we know that:—

- it is easy to handle a system of linear equations when its augmented matrix representation is a reduced row-echelon form, and
- every matrix is row-equivalent to some reduced row-echelon form, which can be obtained methodically, say, with an application of Gaussian elimination.

For this reason, it is very often convenient to choose $A', \mathbf{b}'$ in such a way that $[\ A' \mid \mathbf{b}'\ ]$ is a reduced row echelon form.

(That said, it is not an absolute necessity to do so all the time.)

You have in fact worked in this spirit when 'solving simultaneous linear equations' in school maths despite knowing little of the terminologies here.

Below is an illustration from school maths.

5. **Illustration.**

Solve the simultaneous equations

$$(S): \quad \left\{ \begin{array}{rcrcr} x_1 & + & 3x_2 & = & 3 \\ 2x_1 & - & x_2 & = & 4 \end{array} \right.$$

(a) What we would do in school maths to handle this problem would be to present such a chain of manipulations:—

$$(S_1) \quad \left\{ \begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & = & 3 & \text{——} & (1) \\ x_1 & - & x_2 & = & 4 & \text{——} & (2) \end{array} \right.$$

'Adding (-2 times Equation (2)) to Equation (2)':
$$(S_2) \quad \left\{ \begin{array}{rcrcrcl} & & 5x_2 & = & -5 & \text{——} & (3) \\ x_1 & - & x_2 & = & 4 & \text{——} & (2) \end{array} \right.$$

'Multiplying 1/5 to Equation (3)':
$$(S_3) \quad \left\{ \begin{array}{rcrcrcl} & & x_2 & = & -1 & \text{——} & (4) \\ x_1 & - & x_2 & = & 4 & \text{——} & (2) \end{array} \right.$$

'Adding Equation (4) to Equation (2)':
$$(S_4) \quad \left\{ \begin{array}{rrcrcl} & x_2 & = & -1 & \text{——} & (4) \\ x_1 & & = & 3 & \text{——} & (5) \end{array} \right.$$

'Swapping Equation (4) and Equation (5)':
$$(S') \quad \left\{ \begin{array}{rrcrcl} x_1 & & = & 3 & \text{——} & (5) \\ & x_2 & = & -1 & \text{——} & (4) \end{array} \right.$$

And then we conclude that:—

- *A solution of* (S) *is given by '*$x_1 = 3$ *and* $x_2 = -1$*', and it is the only solution of* (S).

(b) Translated in terms of matrices and vectors, and further presented in terms of augmented matrix representations of various systems, what we have actually done by writing down this chain of manipulations is nothing but presenting the sequence of row operations below, starting with the augmented matrix representation C of (S) and endinig at some reduced row-echelon form C':—

$$C=\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & -1 & 4 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|c} 0 & 5 & -1 \\ 1 & -1 & 4 \end{array}\right] \xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & -1 & 4 \end{array}\right] \xrightarrow{1R_1+R_2} \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} C'=\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

(Seen in this way, the symbols 'x_1', 'x_2', '+', '=' in the chain of manipulations are just book-keeping devices for keeping track of the 'givens' in the various systems presented in the chain of manipulation.)

Then we reason:—

- Theorem (1) says that:—

  $\mathbf{t}$ *is a solution of the system* (S)*, whose augmented matrix representation is* C*,*

  if and only if

  $\mathbf{t}$ *is a solution of the system* (S')*, whose augmented matrix representation is* C'*.*

  Since $\left[\begin{array}{c} 3 \\ -1 \end{array}\right]$ is the only solution of (S'), it is the only solution of (S).

6. Coupled with what we have learnt about row-echelon forms and reduced row-echelon forms, Theorem (1) gives Theorem (2) and Theorem (3).

Theorem (2).

Let A be an $(m \times n)$-matrix, and $\mathbf{b}$ be a column vector with m entries.

Suppose $C = [\ A \mid \mathbf{b}\ ]$, which is the augmented matrix representation of $\mathcal{LS}(A, \mathbf{b})$.

Further suppose $C^\sharp$ is a row-echelon form which is row-equivalent to C.

Write $C^\sharp = [\ A^\sharp \mid \mathbf{b}^\sharp\ ]$, in which $A^\sharp$ stands for the $(m \times n)$-matrix given by the first n columns of $C^\sharp$, and $\mathbf{b}^\sharp$ stands for the last column of $C^\sharp$.

Then $A^\sharp$ is a row-echelon form which is row-equivalent to A.

Moreover, the statements below are logically equivalent:—

(a) *The system $\mathcal{LS}(A, \mathbf{b})$ is consistent.*

(b) *the last column of $C^\sharp$ is a free column.*

(c) *The rank of $A^\sharp$ is equal to the rank of $C^\sharp$.*

Theorem (3).

Let A be an $(m \times n)$-matrix, and $\mathbf{b}$ be a column vector with m entries.

Suppose $C = [\ A \mid \mathbf{b}\ ]$, which is the augmented matrix representation of $\mathcal{LS}(A, \mathbf{b})$.

Further suppose $C^\sharp$ is a row-echelon form which is row-equivalent to C.

Write $C^\sharp = [\ A^\sharp \mid \mathbf{b}^\sharp\ ]$, in which $A^\sharp$ stands for the $(m \times n)$-matrix given by the first n columns of $C^\sharp$, and $\mathbf{b}^\sharp$ stands for the last column of $C^\sharp$.

Suppose $\mathcal{LS}(A, \mathbf{b})$ is consistent. (So the last column of $C^\sharp$ is a free column, and the rank of $A^\sharp$ is equal to the rank of $C^\sharp$.)

(a) *Denote the rank of $C^\sharp$ by r. Then $r \leq n$.*

   Moreover, there are two alternatives:—

   - *(Case 1.) Suppose $r = n$. Then $\mathcal{LS}(A, \mathbf{b})$ has a unique solution.*
   - *(Case 2.) Suppose $r < n$. Then $\mathcal{LS}(A, \mathbf{b})$ has two or more solutions.*

(b) *Further suppose the pivot columns of $C^\sharp$ are, from left to right, the d_1-th, d_2-th, ..., d_r-th columns of $C^\sharp$, with $d_1 = 1$.*

   Then there is some reduced row-echelon form C' of rank r, whose pivot columns, from left to right, are the d_1-th, d_2-th, ..., d_r-th columns of C', such that C' is row-equivalent to C.

(c) *Write $C' = [\ A' \mid \mathbf{b}'\ ]$, in which:*

   - *A' stands for the $(m \times n)$-matrix given by the first n columns of C', and*
   - *$\mathbf{b}'$ stands for the last column of C'.*

   Then:—

- *A' is a reduced row-echelon form of rank r which is row-equivalent to $A^\sharp$, and*
- *the pivot columns of A' are, from left to right, the d_1-th, d_2-th, ..., d_r-th columns of A'.*

(d) *Denote the top r entries of $\mathbf{b}'$, which is the last column of C', by $\beta_1, \beta_2, \cdots, \beta_r$, from the top downwards. Denote by $\mathbf{p}$ the column vectors with n entries, in which:—*

- *the d_1-th, d_2-th, ..., d_r-th entries are $\beta_1, \beta_2, \cdots, \beta_r$ respectively, and*
- *all other entries are 0.*

Then $\mathbf{p}$ is a (particular) solution of $\mathcal{LS}((, A), b)$.

(e) *Suppose $r = n$.*
(So there is no free column in C' other than the last column of C'.)
Then $\mathbf{p}$ is the one and only one solution of $\mathcal{LS}(A, \mathbf{b})$.

(f) *Suppose $r < n$ (instead of supposing '$r = n$').*
Now further suppose the free columns of C', from left to right, are f_1-th, f_2-th, ..., f_{n-r}-th, f_{n+1-r}-th columns.
For each $\ell = 1, 2, \cdots, n-r$, denote the top r entries of the f_ℓ-th column of C' by $\alpha_{1\ell}, \alpha_{2\ell}, \cdots, \alpha_{r\ell}$, from the top downwards.
Further denote by $\mathbf{q}_\ell$ the column vector with n entries, in which:—

- *the d_1-th, d_2-th, ..., d_r-th entries are $-\alpha_{1\ell}, -\alpha_{2\ell}, \cdots, -\alpha_{r\ell}$,*
- *the f_ℓ-th entry is 1, and*
- *all other entries are 0.*

Then the statements below hold:—

i. *The column vectors $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{n-r}$ are linearly independent.*
ii. *Suppose $\mathbf{t}$ is a column vector with n entries.*
Then $\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if
there are some numbers $u_1, u_2, \cdots, u_{n-r}$ such that $\mathbf{t} = \mathbf{p} + u_1\mathbf{q}_1 + u_2\mathbf{q}_2 + \cdots + u_{n-r}\mathbf{q}_{n-r}$.

7. **Example (1). (How to solve a system of linear equations, through finding a reduced row-echelon form, as suggested by Theorem (2) and Theorem (3).)**

(a) *We solve the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, in which $A, \mathbf{b}$ are given by*

$$A = \begin{bmatrix} 1 & -1 & 2 & -7 \\ 3 & -2 & 6 & -18 \\ -4 & 3 & -7 & 23 \\ 1 & 2 & 0 & 7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -23 \\ -55 \\ 73 \\ 33 \end{bmatrix}.$$

The augmented matrix representation C of $\mathcal{LS}(A, \mathbf{b})$ is given by

$$C = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -7 & -23 \\ 3 & -2 & 6 & -18 & -55 \\ -4 & 3 & -7 & 23 & 73 \\ 1 & 2 & 0 & 7 & 33 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ and a reduced row-echelon form C' which are row-equivalent to C, through the sequence of row operations below:

$$C \xrightarrow{-3R_1+R_2} \xrightarrow{4R_1+R_3} \xrightarrow{-1R_1+R_4} \xrightarrow{1R_2+R_3} \xrightarrow{-3R_2+R_4} \xrightarrow{2R_3+R_4} C^\sharp \xrightarrow{1R_2+R_1} \xrightarrow{-2R_3+R_1} \xrightarrow{-3R_4+R_2} \xrightarrow{2R_4+R_3} C'$$

Note that $C' = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$, and it is the augmented matrix representation for the system $\mathcal{LS}(A', \mathbf{b}')$, in which

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b}' = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(Note that $\mathcal{LS}(A', \mathbf{b}')$ reads

$$\begin{cases} x_1 & & & & = & 1 \\ & x_2 & & & = & 2 \\ & & x_3 & & = & 3 \\ & & & x_4 & = & 4 \end{cases}$$

when it is written out explicitly.)
It follows that:—

the one and only one solution of $\mathcal{LS}(A,\,\mathbf{b})$ *is* $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(b) *We solve the system of linear equations* $\mathcal{LS}(A,\,\mathbf{b})$, *in which* $A, \mathbf{b}$ *are given by*

$$A = \begin{bmatrix} 1 & 3 & -2 & 3 & 21 \\ 2 & 6 & -3 & 5 & 38 \\ 1 & 3 & -4 & 6 & 33 \\ -2 & -6 & 3 & -6 & -42 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The augmented matrix representation C of $\mathcal{LS}(A,\,\mathbf{b})$ is given by

$$C = \left[\begin{array}{ccccc|c} 1 & 3 & -2 & 3 & 21 & 0 \\ 2 & 6 & -3 & 5 & 38 & 0 \\ 1 & 3 & -4 & 6 & 33 & 0 \\ -2 & -6 & 3 & -6 & -42 & 1 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ which is row-equivalent to C, through the sequence of row operations below:

$$C \xrightarrow{-2R_1+R_2} \xrightarrow{-1R_1+R_3} \xrightarrow{2R_1+R_4} \xrightarrow{2R_2+R_3} \xrightarrow{1R_2+R_4} \xrightarrow{1R_3+R_4} C^\sharp$$

Note that $C^\sharp = \left[\begin{array}{ccccc|c} 1 & 3 & -2 & 3 & 21 & 0 \\ 0 & 0 & 1 & -1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$, and it is the augmented matrix representation for the system $\mathcal{LS}\big(A^\sharp,\,\mathbf{b}^\sharp\big)$, in which

$$A^\sharp = \begin{bmatrix} 1 & 3 & -2 & 3 & 21 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{b}^\sharp = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(Note that $\mathcal{LS}\big(A^\sharp,\,\mathbf{b}^\sharp\big)$ reads

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 3x_4 + 21x_5 = 0 \\ x_3 - x_4 - 4x_5 = 0 \\ x_4 + 4x_5 = 0 \\ 0 = 1 \end{cases}$$

when it is written out explicitly.)
It follows that:—

$\mathcal{LS}(A,\,\mathbf{b})$ has no solution.

(c) *We solve the system of linear equations* $\mathcal{LS}(A,\,\mathbf{b})$, *in which* $A, \mathbf{b}$ *are given by*

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 \\ 1 & 3 & 2 & -3 & -3 \\ 2 & 6 & 1 & -2 & 10 \\ -1 & -3 & -3 & 1 & -5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -3 \\ -4 \\ -3 \\ -1 \end{bmatrix}.$$

The augmented matrix representation C of $\mathcal{LS}(A,\,\mathbf{b})$ is given by

$$C = \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -2 & 1 & -3 \\ 1 & 3 & 2 & -3 & -3 & -4 \\ 2 & 6 & 1 & -2 & 10 & -3 \\ -1 & -3 & -3 & 1 & -5 & -1 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ and a reduced row-echelon form C' which are row-equivalent to C, through the sequence of row operations below:

$$C \xrightarrow{-1R_1+R_2} \xrightarrow{-2R_1+R_3} \xrightarrow{1R_1+R_4} \xrightarrow{1R_2+R_3} \xrightarrow{2R_2+R_4} \xrightarrow{3R_3+R_4} C^\sharp \xrightarrow{-1R_2+R_1} \xrightarrow{1R_3+R_1} \xrightarrow{1R_3+R_2} C'$$

Note that $C' = \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$, and it is the augmented matrix representation for the system $\mathcal{LS}(A',\,\mathbf{b}')$, in which

$$A' = \begin{bmatrix} 1 & 3 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{b}' = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

(Note that $\mathcal{LS}(A', \mathbf{b}')$ reads

$$\begin{cases} x_1 + 3x_2 + 9x_5 = 0 \\ x_3 = 1 \\ x_4 + 4x_5 = 2 \\ 0 = 0 \end{cases},$$

or equivalently,

$$\begin{cases} x_1 = -3x_2 - 9x_5 \\ x_3 = 1 \\ x_4 = 2 - 4x_5 \end{cases}$$

when it is written out explicitly.)

It follows that a full description of all solutions of $\mathcal{LS}(A, \mathbf{b})$ is given by:—

- *$\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if*

$$\text{there are some numbers } u, v \text{ such that } \mathbf{t} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} + u \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -9 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

(d) *We solve the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, in which $A, \mathbf{b}$ are given by*

$$A = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 \\ -1 & 2 & 1 & -1 & 0 & -2 \\ 2 & -4 & -1 & 3 & 2 & 1 \\ 3 & -6 & -1 & 5 & 4 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}.$$

The augmented matrix representation C of $\mathcal{LS}(A, \mathbf{b})$ is given by

$$C = \left[\begin{array}{cccccc|c} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{array}\right].$$

We obtain a row-echelon form $C^\sharp$ and a reduced row-echelon form C' which are row-equivalent to C, through the sequence of row operations below:

$$C \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{-1R_5} \xrightarrow{-2R_1+R_3} \xrightarrow{-3R_1+R_4} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{-2R_2+R_3} \xrightarrow{-2R_2+R_4} C^\sharp \xrightarrow{1R_2+R_1} \xrightarrow{-2R_3+R_1} \xrightarrow{-1R_3+R_2} C'$$

Note that $C' = \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$, and it is the augmented matrix representation for the system $\mathcal{LS}(A', \mathbf{b}')$, in which

$$A' = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{b}' = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}$$

(Note that $\mathcal{LS}(A', \mathbf{b}')$ reads

$$\begin{cases} x_1 - 2x_2 + x_6 = 1 \\ x_3 + x_5 - 2x_6 = 3 \\ x_4 + x_5 - x_6 = 2 \\ 0 = 0 \end{cases},$$

or equivalently,

$$\begin{cases} x_1 = 1 + 2x_2 - x_6 \\ x_3 = 3 - x_5 + 2x_6 \\ x_4 = 2 - x_5 + x_6 \end{cases}$$

when it is written out explicitly.)

It follows that a full description of all solutions of $\mathcal{LS}(A, \mathbf{b})$ is given by:—

- *$\mathbf{t}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if* *there are some numbers u, v, w such that* $\mathbf{t} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

8. **Example (2). (How to solve a homogeneous system of linear equations, through finding a reduced row-echelon form, as suggested by Theorem (2) and Theorem (3).)**

   Remember:—

   - *Every homogeneous system is consistent, with a trivial solution* $\mathbf{0}$. (The question is whether it has any non-trivial solution.)
   - *Suppose A, A' are row-equivalent $(m \times n)$-matrices. Then* $[\ A \mid \mathbf{0}_m\ ], [\ A' \mid \mathbf{0}_m\ ]$ *are row-equivalent. Moreover, if A' is a (reduced) row-echelon form, then* $[\ A' \mid \mathbf{0}_m\ ]$ *is a (reduced) row-echelon form.*

   (a) *We want to solve the homogeneous system of linear equations* $\mathcal{LS}(A, \mathbf{0}_5)$, *in which*

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & -2 & -6 & 3 & 8 \\ -2 & -4 & 3 & -5 & 6 & 28 & -9 & -18 \\ 1 & 2 & -2 & 4 & -4 & -15 & 7 & 19 \\ -3 & -6 & 5 & -6 & 11 & 73 & -14 & -5 \\ -1 & -2 & 2 & -5 & 4 & 8 & -7 & -27 \end{bmatrix}.$$

   We obtain a reduced row-echelon form A' which are row-equivalent to A, through the sequence of row operations below:

$$A \xrightarrow{2R_1+R_2} \xrightarrow{-1R_1+R_3} \xrightarrow{3R_1+R_4} \xrightarrow{1R_1+R_5} \xrightarrow{1R_2+R_3} \xrightarrow{-2R_2+R_4} \xrightarrow{-1R_2+R_5} \xrightarrow{-2R_3+R_4} \xrightarrow{2R_3+R_5}$$
$$\xrightarrow{1R_2+R_1} \xrightarrow{-1R_3+R_1} \xrightarrow{1R_3+R_2} \xrightarrow{-2R_4+R_2} \xrightarrow{1R_5+R_1} \xrightarrow{-1R_5+R_3} \xrightarrow{1R_5+R_4} A'$$

   Note that $A' = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 7 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 9 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$.

   (Note that $\mathcal{LS}(A', \mathbf{0}_5)$ reads

$$\left\{ \begin{array}{rrrrrrrrrrrrl} x_1 & + & 2x_2 & & & & & + & 3x_6 & & - & 2x_8 & = 0 \\ & & & x_3 & & & & + & 5x_6 & & - & 3x_8 & = 0 \\ & & & & x_4 & & & + & 7x_6 & & + & 8x_8 & = 0 \\ & & & & & x_5 & & + & 9x_6 & & + & 6x_8 & = 0 \\ & & & & & & & & & x_7 & + & 4x_8 & = 0 \end{array} \right.$$

   or equivalently

$$\left\{ \begin{array}{rcrrr} x_1 & = & -2x_2 & -3x_6 & +2x_8 \\ x_3 & = & & -5x_6 & +3x_8 \\ x_4 & = & & -7x_6 & -8x_8 \\ x_5 & = & & -9x_6 & -6x_8 \\ x_7 & = & & & -4x_8 \end{array} \right.$$

   when it is written out explicitly.)

   It follows that a full description of all solutions of $\mathcal{LS}(A, \mathbf{0}_5)$ is given by:—

   - $\mathbf{t}$ *is a solution of* $\mathcal{LS}(A, \mathbf{0}_5)$ *if and only if*

     there are some numbers u, v, w such that $\mathbf{t} = u \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 0 \\ -5 \\ -7 \\ -9 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ 0 \\ 3 \\ -8 \\ -6 \\ 0 \\ 1 \\ -4 \end{bmatrix}$.

   (b) *We want to solve the homogeneous system of linear equations* $\mathcal{LS}(A, \mathbf{0}_5)$, *in which*

$$A = \begin{bmatrix} 1 & 2 & -5 & 15 \\ -1 & -1 & 3 & -9 \\ 3 & 4 & -10 & 31 \\ 2 & 3 & -8 & 25 \\ 1 & 3 & -4 & 13 \end{bmatrix}.$$

We obtain a reduced row-echelon form A' which are row-equivalent to A, through the sequence of row operations below:

$$A \xrightarrow{1R_1+R_2} \xrightarrow{-3R_1+R_3} \xrightarrow{-2R_1+R_4} \xrightarrow{-1R_1+R_5} \xrightarrow{2R_2+R_3} \xrightarrow{1R_2+R_4} \xrightarrow{-1R_2+R_5} \xrightarrow{-3R_3+R_5} \xrightarrow{2R_4+R_5}$$
$$\xrightarrow{-2R_2+R_1} \xrightarrow{1R_3+R_1} \xrightarrow{2R_3+R_2} \xrightarrow{-1R_4+R_1} \xrightarrow{-2R_4+R_2} \xrightarrow{2R_4+R_3} A'$$

Note that $A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(Note that $\mathcal{LS}(A', \mathbf{0}_5)$ reads

$$\left\{ \begin{array}{ccccccc} x_1 & & & & & = & 0 \\ & & x_2 & & & = & 0 \\ & & & x_3 & & = & 0 \\ & & & & x_4 & = & 0 \\ & & & & & = & 0 \end{array} \right.$$

when it is written out explicitly.)

It follows that:—

- *the one and only one solution of $\mathcal{LS}(A, \mathbf{0}_5)$ is the trivial solution $\mathbf{0}_4$.*

9. We state a result which relates the number of rows of the coefficient matrix of a consistent system of linear equations and the rank of a row-echelon form of the coefficient matrix. It is an immediate consequence of Theorem (2) and Theorem (3).

Theorem (4).

Suppose A is an $(m \times n)$-matrix, and r is the rank of a row-echelon form $A^\sharp$ which is row-equivalent to A.

Then the statements below hold:

(a) *$r \leq m$ and $r \leq n$.*

(b) *Suppose $m < n$. Then:—*

  i. *For any column vector $\mathbf{b}$ with m entries, if the system $\mathcal{LS}(A, \mathbf{b})$ is consistent, then $\mathcal{LS}(A, \mathbf{b})$ has two or more solutions.*

  ii. *If $r = m$, then for any column vector $\mathbf{b}$ with m entries, the system $\mathcal{LS}(A, \mathbf{b})$ is consistent and has two or more solutions.*

  iii. *The homogeneous system $\mathcal{LS}(A, \mathbf{0}_m)$ has a non-trivial solution.*

Remark. In plain words (about equations), part (b.i) the conclusion can be expressed as:—

'*If the number of (linear) equations in a system of linear equations is less than the number of unknowns, then either there is no solution, or there are two or more solutions.*'

In order for a consistent system of linear equations to have one and only one solution, it is necessary for the number of equations to be at least as many as the number of unknowns.

This idea is a cornerstone in many areas of mathematics.

10. **Example (3). ('Baby examples' from school maths that illustrates Theorem (4).)**

(a) Let $A = [\ 1 \quad 2\ ]$. Note that A is a row-echelon form.

For each number b, the system of linear equations $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [b]$, which is the equation $x_1 + 2x_2 = b$ in disguise, has two or more solution.

(b) Let $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$. A row-echelon form $A^\sharp$ which is row-equivalent to A is given by $A^\sharp = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

For any numbers b_1, b_2, the system of linear equations $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ either has no solution, or has two or more solution.

When it is written out explicitly, the system reads:—

$$\left\{ \begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & b_1 \\ 2x_1 & + & 4x_2 & + & 6x_3 & = & b_2 \end{array} \right.$$

It is a system with two equations and three unknowns.

- The system has no solution if and only if $b_2 \neq 2b_1$.
- The system has two or more solutions if and only if $b_2 = 2b_1$.

(c) Let $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. A row-echelon form $A^\sharp$ which is row-equivalent to A is given by $A^\sharp = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

For any numbers b_1, b_2, the system of linear equations $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ either has no solution, or has two or more solution.
When it is written out explicitly, the system reads:—

$$\begin{cases} x_1 + x_2 + x_3 = b_1 \\ \quad\;\; x_2 + x_3 = b_2 \end{cases}$$

It is a system with two equations and three unknowns.
The system is consistent and has infinitely many solutions.

11. **Example (4). (Illustrations on Theorem (4).)**

(a) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$.

Note that a row-echelon form $A^\sharp$ which is row equivalent to A is given by $A^\sharp = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Given any column vector $\mathbf{b}$ with 4 entries, it will happen that:—

- either $\mathcal{LS}(A, \mathbf{b})$ is inconsistent,
- or $\mathcal{LS}(A, \mathbf{b})$ is consistent and has two or more solutions.

In fact:—

- when $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathcal{LS}(A, \mathbf{c})$ is inconsistent.
- $\mathcal{LS}(A, \mathbf{0}_4)$ is consistent and has two or more solutions.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \end{bmatrix}$.

Note that a row-echelon form $A^\sharp$ which is row equivalent to A is given by $A^\sharp = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$.

Given any column vector $\mathbf{b}$ with 4 entries, it will happen that $\mathcal{LS}(A, \mathbf{b})$ is consistent and has two or more solutions.