2.3 Existence of reduced row-echelon form row-equivalent to given matrix, (and the uniqueness question).

0. *Assumed background.*
   - 1.7 *Row operations on matrices.*
   - 1.8 *Row operations and matrix multiplication.*
   - 2.2 *Row-echelon forms and reduced row-echelon forms.*

   Abstract. We introduce:—

   - Theorem of existence of row-echelon form which is row-equivalent to a given matrix.
   - Theorem of existence and uniqueness of reduced row-echelon form which is row-equivalent to a given matrix.
   - Gaussian elimination, as an 'algorithm' for finding in a systematic way a row-echelon form, and a reduced row-echelon form, which are row-equivalent to a given matrix.

   The proofs of the two existence theorems are contained in the *appendix*, but the ideas in the proofs are displayed in the application of Gaussian elimination in every 'concrete' example.

1. Row-echelon forms and reduced row-echelon forms are special types of matrices. They are interesting because of the fundamental theoretical results introduced below which relate arbitrary matrices to row-echelon forms and reduced row-echelon forms through row operations.

   Theorem (1). (Existence of row-echelon form which is row-equivalent to a given matrix.)

   Suppose that C is a matrix.

   Then there exists some row-echelon form $C^\sharp$ such that $C^\sharp$ is row-equivalent to C.

   Theorem (2). (Existence and uniqueness of reduced row-echelon form which is row-equivalent to a given matrix.)

   Suppose that C is a matrix.

   Then there exists some unique reduced row-echelon form C' such that C' is row-equivalent to C.

   Remark. Theorem (2), which concerned with reduced row echelon forms, is an *existence-and-uniqueness result.* It is a combination of two results, Theorem (3) and Theorem (4), which are logically independent of each other.

2. **Theorem (3). (Existence of reduced row-echelon form which is row-equivalent to a given matrix.)**

   Suppose that C is a matrix.

   Then there exists some reduced row-echelon form C' such that C' is row-equivalent to C.

3. Using the 'dictionary' between application of row-operations and left-multiplication by row-operation matrices, we have the logically equivalent formulation of Theorem (3) below:—

   Corollary to Theorem (3). ('Factorization' of a given matrix as a product of row-operation matrices multiplied from the left to a reduced row-echelon form.)

   Suppose that C is a matrix.

   Then there exist some reduced row-echelon form C', and some square matrix H, such that the equality $C' = HC$ holds and H is a product of row-operation matrices.

4. **Comments on Theorem (1) and Theorem (3).**

   Theorem (1) and Theorem (3) are useful devices in many theoretical discussions in this course. They will be used very often.

   An outline of the argument for these results will be given later. The nature of the argument is a 'constructive' argument.

   The idea in the argument is displayed when we apply the idea in the concrete situation for systematically finding a row-echelon form or a reduced row-echelon form which is row-equivalent to any arbitrarily given matrix.

   This process (or 'algorithm') is known as **Gaussian elimination**.

5. **Theorem (4). (Uniqueness of reduced row-echelon form which is row-equivalent to a given matrix.)**

   Suppose that C is a matrix, and C', C'' are reduced row-echelon forms.

   Further suppose that C is row-equivalent to C', and C is also row-equivalent to C''.

   Then $C' = C''$.

   Remark. The proof of Theorem (4) is omitted for now.

6. Because of the basic properties of row-equivalence, Theorem (4) is logically equivalent to the result below:—

   Corollary to Theorem (4). (Uniqueness of reduced row-echelon form which is row-equivalent to each other.)

   Let D, E be reduced row-echelon forms. Suppose that D is row-equivalent to E.

   Then $D = E$.

7. **Comments on Theorem (2), Theorem (3) and Theorem (4).**

   Taking into account the basic properties of row equivalence, and also regarding Theorem (2), Theorem (3) and Theorem (4) as valid, we will obtain the picture on the 'world' of all $(p \times q)$-matrices below:—

   - Row-equivalence partitions the 'world' of all $(p \times q)$-matrices into 'chambers' of various $(p \times q)$-matrices.
   - The matrices within each 'chamber' are row-equivalent to each other.
   - No matrix in a 'chamber' will be row-equivalent to any matrix in any distinct 'chamber'.
   - Within each chamber there is one and only one reduced row-echelon form (which is row-equivalent to every other matrix in that 'chamber').

8. **Example (1). (Idea of Gaussian elimination, introduced through a concrete example.)**

   Let $C = \begin{bmatrix} 0 & -1 & -2 & 2 & 6 \\ 2 & -3 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 1 \\ 2 & -4 & -2 & 2 & 6 \end{bmatrix}$.

   We apply a sequence of row operations on C to:—

   - *first obtain a row-echelon form $C^\sharp$ which is row-equivalent to C, and*
   - *then obtain a reduced row-echelon form C' which is row-equivalent to C.*

   The systematic way that we are applying row operations to obtain $C^\sharp, C'$ from C is referred to as **Gaussian elimination**.

   (a) Apply row operations of the types '$\alpha R_i + R_k$', '$R_i \leftrightarrow R_k$' on C to obtain some matrix C_1 in which:—

   - the first non-zero entry in the first row is strictly to the left of the first non-zero entry in every row below the first row.

   Optional: apply a row operation of the type 'βR_k' to make 1 the first non-zero entry of the top row in C_1.

$$C = \begin{bmatrix} 0 & -1 & -2 & 2 & 6 \\ 2 & -3 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 1 \\ 2 & -4 & -2 & 2 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 2 & -4 & -2 & 2 & 6 \\ 2 & -3 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 1 \\ 0 & -1 & -2 & 2 & 6 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & -1 & 1 & 3 \\ 2 & -3 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 1 \\ 0 & -1 & -2 & 2 & 6 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & -2 & -1 & 1 & 3 \\ 0 & 1 & 2 & -2 & -6 \\ -2 & 2 & -2 & 1 & 1 \\ 0 & -1 & -2 & 2 & 6 \end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & -2 & -1 & 1 & 3 \\ 0 & 1 & 2 & -2 & -6 \\ 0 & -2 & -4 & 3 & 7 \\ 0 & -1 & -2 & 2 & 6 \end{bmatrix} = C_1$$

   (b) Inspect C_1 to see whether it is a row-echelon form.

   If not, apply an appropriate sequence of row operations of the types '$\alpha R_i + R_k$', '$R_i \leftrightarrow R_k$' on C_1 to obtain some matrix C_2 in which:—

   - the first non-zero entry in the second row is strictly to the left of the first non-zero entry in every row below the second row.

   (c) Note that C_1 is indeed not a row-echelon form.

$$C_1 = \begin{bmatrix} 1 & -2 & -1 & 1 & 3 \\ 0 & 1 & 2 & -2 & -6 \\ 0 & -2 & -4 & 3 & 7 \\ 0 & -1 & -2 & 2 & 6 \end{bmatrix} \xrightarrow{1R_2+R_4} \begin{bmatrix} 1 & -2 & -1 & 1 & 3 \\ 0 & 1 & 2 & -2 & -6 \\ 0 & -2 & -4 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_2+R_3} \begin{bmatrix} 1 & -2 & -1 & 1 & 3 \\ 0 & 1 & 2 & -2 & -6 \\ 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C_2$$

   (d) Inspect C_2 to see whether it is a row-echelon form. If not, we apply an appropriate sequence of row operations of the types '$\alpha R_i + R_k$', '$R_i \leftrightarrow R_k$' on C_2 to obtain some matrix C_3 in which:—

   - the first non-zero entry in the third row is strictly to the left of the first non-zero entry in every row below the third row.

   Et cetera.

(e) C_2 is a row-echelon form, and we re-label C_2 as $C^\sharp$.

Apply row operations of the types '$\alpha R_i + R_k$', 'βR_k' on $C^\sharp$ to obtain C', which is a reduced row-echelon form:

$$C^\sharp = \begin{bmatrix} 1 & -2 & -1 & 1 & 3 \\ 0 & 1 & 2 & -2 & -6 \\ 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_3} \begin{bmatrix} 1 & -2 & -1 & 1 & 3 \\ 0 & 1 & 2 & -2 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{2R_2+R_1} \begin{bmatrix} 1 & 0 & 3 & -3 & -9 \\ 0 & 1 & 2 & -2 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{3R_3+R_1} \begin{bmatrix} 1 & 0 & 3 & 0 & 6 \\ 0 & 1 & 2 & -2 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_3+R_2} \begin{bmatrix} 1 & 0 & 3 & 0 & 6 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C'$$

(f) By construction, C' is a reduced row-echelon form which is row-equivalent to C.

As a bonus, we obtain the equality $C' = HC$, in which H is the resultant of the application on I_4 of the sequence of row operations below (with which we obtain C' from C):

$$I_4 \xrightarrow{R_1 \leftrightarrow R_4} \xrightarrow{\frac{1}{2}R_1} \xrightarrow{-2R_1+R_2} \xrightarrow{2R_1+R_3} \xrightarrow{1R_2+R_4} \xrightarrow{2R_2+R_3} \xrightarrow{-1R_3} \xrightarrow{2R_2+R_1} \xrightarrow{3R_3+R_1} \xrightarrow{2R_3+R_2} H$$

We have $H = \begin{bmatrix} 0 & -4 & -3 & 3/2 \\ 0 & -3 & -2 & 1 \\ 0 & -2 & -1 & 1 \\ 1 & 1 & 0 & -1 \end{bmatrix}$.

9. **Example (2). (Another concrete example on Gaussian elimination.)**

Let $C = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$.

(a) Note that C is not a row-echelon form.

Apply row operations of the types '$\alpha R_i + R_k$', '$R_i \leftrightarrow R_k$' on C to obtain some matrix C_1 in which:—

- the first non-zero entry in the top row is strictly to the left of the first non-zero entry in every other row.

$$C = \begin{bmatrix} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$$

$$\xrightarrow{-1R_1} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$$

$$\xrightarrow{-3R_1+R_4} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 2 & 2 & 4 & -6 & 10 \end{bmatrix} = C_1$$

(b) Note that C_1 is not a row-echelon form.

Apply an appropriate sequence of row operations of the types '$\alpha R_i + R_k$', '$R_i \leftrightarrow R_k$' on C_1 to obtain some matrix C_2 in which:—

- the first non-zero entry in the second row is strictly to the left of the first non-zero entry in every row below the second row.

$$C_1 = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 2 & 2 & 4 & -6 & 10 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 0 & 0 & 2 & 2 & 4 & -6 & 10 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 2 & 4 & -6 & 10 \end{bmatrix} \xrightarrow{-2R_2+R_4} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C_2$$

(c) Note that C_2 is a row-echelon form. Write C_2 as $C^\sharp$.

Apply row operations of the types '$\alpha R_i + R_k$', 'βR_k' on $C^\sharp$ to obtain C', which is a reduced row-echelon form:

$$C^\sharp = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{1R_2+R_1} \begin{bmatrix} 1 & -2 & 0 & 2 & 2 & -1 & 5 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_3+R_2} \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C'$$

(d) By construction, C' is a reduced row-echelon form which is row-equivalent to C.

As a bonus, we obtain the equality $C' = HC$, in which H is the resultant of the application on I_4 of the sequence of row operations below (with which we obtain C' from C):

$$I_4 \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{-1R_1} \xrightarrow{-2R_1+R_3} \xrightarrow{-3R_1+R_4} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{-2R_2+R_3} \xrightarrow{-2R_2+R_4} \xrightarrow{1R_2+R_1} \xrightarrow{-2R_3+R_1} \xrightarrow{-1R_3+R_2} H$$

We have $H = \begin{bmatrix} -2 & 9 & 5 & 0 \\ -1 & 6 & 3 & 0 \\ 1 & -4 & -2 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}$.

10. **Gaussian Elimination as an algorithm.**

Suppose C is a matrix with p rows.

We describe how to systematically obtain a row-echelon form $C^\sharp$ and then a reduced row-echelon form C' which are row-equivalent to C:—

Step (0). Write $C_0 = C$. Inspect C_0. Ask:—

Is C_0 a row-echelon form?

- If *no*, go to Step (1).
- If *yes*, write $C^\sharp = C_0$ and go to Step (2).

Step (1). Suppose $k = 1, 2, \cdots, p$.

(a) If C_{k-1} is not a row-echelon form, then apply a sequence of row operations on C_{k-1} to obtain some matrix C_k in which:—
  - the first non-zero entry in the k-th row of C_k is strictly to the left of the first non-zero entry in every row below the k-th row.

(b) Next inspect C_k. Ask:—

Is C_k a row-echelon form?
  - If *no*, iterate as described above to obtain C_{k+1}.
  - If *yes*, write $C^\sharp = C_k$ and go to Step (2).

Step (2). Inspect $C^\sharp$.

If $C^\sharp$ is not already a reduced row-echelon form, then apply a sequence of row operations on $C^\sharp$ to obtain some reduced row-echelon form C'.

11. **Example (3). (More illustrations on Gaussian elimination.)**

(a) Let $C = \begin{bmatrix} 2 & 7 & -8 \\ 1 & 4 & -5 \\ -1 & -1 & 1 \\ -2 & -6 & 6 \end{bmatrix}$.

We apply row operations on C to obtain some row-echelon form $C^\sharp$ which is row-equivalent to C:—

$$C = \begin{bmatrix} 2 & 7 & -8 \\ 1 & 4 & -5 \\ -1 & -1 & 1 \\ -2 & -6 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & -5 \\ 2 & 7 & -8 \\ -1 & -1 & 1 \\ -2 & -6 & 6 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 4 & -5 \\ 0 & -1 & 2 \\ -1 & -1 & 1 \\ -2 & -6 & 6 \end{bmatrix} \xrightarrow{1R_1+R_3} \begin{bmatrix} 1 & 4 & -5 \\ 0 & -1 & 2 \\ 0 & 3 & -4 \\ -2 & -6 & 6 \end{bmatrix}$$

$$\xrightarrow{2R_1+R_4} \begin{bmatrix} 1 & 4 & -5 \\ 0 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 2 & -4 \end{bmatrix} \xrightarrow{3R_2+R_3} \begin{bmatrix} 1 & 4 & -5 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & -4 \end{bmatrix} \xrightarrow{2R_2+R_4} \begin{bmatrix} 1 & 4 & -5 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = C^\sharp$$

We apply further row operations on $C^\sharp$ to obtain a reduced row-echelon form C' which is row-equivalent to $C^\sharp$ (and hence C as well):—

$$C^\sharp = \begin{bmatrix} 1 & 4 & -5 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_2} \begin{bmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-4R_2+R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-3R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = C'$$

We also obtain the equality $C' = HC$, in which H is the resultant of the application on I_4 of the sequence of row operations below (with which we obtain C' from C):

$$I_4 \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{-2R_1+R_2} \xrightarrow{1R_1+R_3} \xrightarrow{2R_1+R_4} \xrightarrow{3R_2+R_3} \xrightarrow{2R_2+R_4} \xrightarrow{-1R_2} \xrightarrow{\frac{1}{2}R_3} \xrightarrow{-4R_2+R_1} \xrightarrow{-3R_3+R_1} \xrightarrow{2R_3+R_2} H$$

We have $H = \begin{bmatrix} -1/2 & 1/2 & -3/2 & 0 \\ 2 & -3 & 1 & 0 \\ 3/2 & -5/2 & 1/2 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix}$.

(b) Let $C = \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 2 & 2 & 2 & -7 & -8 & -3 \\ 3 & 2 & 1 & -5 & -7 & 5 \\ 2 & 4 & 6 & -4 & -9 & 2 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{bmatrix}$.

We apply row operations on C to obtain some row-echelon form $C^\sharp$ which is row-equivalent to C:—

$$C = \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 2 & 2 & 2 & -7 & -8 & -3 \\ 3 & 2 & 1 & -5 & -7 & 5 \\ 2 & 4 & 6 & -4 & -9 & 2 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & -3 & -2 & -5 \\ 3 & 2 & 1 & -5 & -7 & 5 \\ 2 & 4 & 6 & -4 & -9 & 2 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & -3 & -2 & -5 \\ 0 & -1 & -2 & 1 & 2 & 2 \\ 2 & 4 & 6 & -4 & -9 & 2 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_4} \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & -3 & -2 & -5 \\ 0 & -1 & -2 & 1 & 2 & 2 \\ 0 & 2 & 4 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_5} \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & -1 & -2 & 1 & 2 & 2 \\ 0 & 2 & 4 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 & -2 & -5 \end{bmatrix}$$

$$\xrightarrow{1R_2+R_3} \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 2 & 4 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 & -2 & -5 \end{bmatrix} \xrightarrow{-2R_2+R_4} \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 & -2 & -5 \end{bmatrix}$$

$$\xrightarrow{3R_3+R_5} \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{2R_4+R_5} \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C^\sharp$$

We apply further row operations on $C^\sharp$ to obtain a reduced row-echelon form C' which is row-equivalent to $C^\sharp$ (and hence C as well):—

$$C^\sharp = \begin{bmatrix} 1 & 1 & 1 & -2 & -3 & 1 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_3+R_1} \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 6 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{1R_4+R_1} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 4 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_4+R_2} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 4 \\ 0 & 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C'$$

We also obtain the equality $C' = HC$, in which H is the resultant of the application on I_5 of the sequence of row operations below (with which we obtain C' from C):

$$I_5 \xrightarrow{-2R_1+R_2} \xrightarrow{-3R_1+R_3} \xrightarrow{-2R_1+R_4} \xrightarrow{R_2 \leftrightarrow R_5} \xrightarrow{1R_2+R_3} \xrightarrow{-2R_2+R_4} \xrightarrow{3R_3+R_5} \xrightarrow{2R_4+R_5} \xrightarrow{-1R_2+R_1} \xrightarrow{2R_3+R_1} \xrightarrow{1R_4+R_1} \xrightarrow{2R_4+R_2} H$$

We have $H = \begin{bmatrix} -7 & 0 & 2 & 1 & -1 \\ -4 & 0 & 0 & 2 & -3 \\ -3 & 0 & 1 & 0 & 1 \\ -2 & 0 & 0 & 1 & -2 \\ -15 & 1 & 3 & 2 & -1 \end{bmatrix}$.

12. The examples on Gaussian elimination that we have seen above suggest Lemma (5) and Theorem (6). The former will turn out to be a preparatory step for the proof of Theorem (3), while the latter will turn out to be a by-product of the proof of Theorem (6). Their proofs are given alongside that of Theorem (3).

    Lemma (5).

    For any natural number r, if A is a row-echelon form of rank r, there exists some reduced row-echelon form A' of rank r such that A' is row-equivalent to A.

    Theorem (6).

    Suppose A is a row-echelon form of rank r, whose pivot columns are, say, the d_1-th, d_2-th, ..., d_r-th columns of A.

    Then there is some reduced row-echelon form A' of rank r, whose pivot columns are the d_1-th, d_2-th, ..., d_r-th columns of A', so that A' is row-equivalent to A.

13. We never talk about '*the row-echelon form* which is row-equivalent to a given matrix'. The reason is that:—

    - given any matrix, there are in general many row-echelon forms which are row-equivalent to the matrix concerned.

    Illustration on the non-uniqueness of row-echelon forms which are row-equivant to a given matrix.

    Let $C_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $C_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $C_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

    Note that C_1, C_2, C_3, C_4 are row-echelon forms which are row-equivalent to each other. In fact, there are many more row-echelon forms which are row-equivalent to them.