1.6 Linear dependence and linear independence.

0. *Assumed background.*

   - 1.5 *Linear combinations.*
   - 2.1 *Systems of linear equations.*

Abstract. We introduce:—

- the notion of linear dependence and that of linear independence, and how they are related at the logical level,
- how the notion of linear dependence and that of linear independence are re-formulated in terms of homogeneous systems of linear equations,
- how the notion of linear dependence and that of linear independence are re-formulated in terms of linear combinations.

1. **Definition. (Linear dependence and linear independence for column/row vectors over real (or complex) numbers.)**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *be column/row vectors with* p *real (or complex) entries. (These* q *vectors are not assumed to be pairwise distinct.)*

   (a) *We say that* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are* **linearly dependent over the real (or complex) numbers** *if and only if the statement* (LD) *holds:*

   (LD) *There exist some real (or complex) numbers* $\alpha_1, \alpha_2, \cdots, \alpha_q$ *such that* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_q$ *are not all zero.*

   The equality $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ *in which* $\alpha_1, \alpha_2, \cdots, \alpha_q$ *are not all zero is called a* **non-trivial linear relation** *of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$*.*

   (b) *We say that* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are* **linearly independent over the real (or complex) numbers** *if and only if the statement* (LI) *holds:*

   (LI) *For any real (or complex) numbers* $\alpha_1, \alpha_2, \cdots, \alpha_q$*, if* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_q = 0$*.*

   Remarks on terminologies.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are the j_1-th, j_2-th, ..., j_q-th columns/rows in a matrix, say, A, with real (or complex) entries.

   Then we say the j_1-th, j_2-th, ..., j_q-th columns/rows of A are linearly dependent//independent over the real (or complex) numbers if and only if $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent//independent over the real (or complex) numbers.

   Furthermore, if A is a $(p \times q)$-matrix and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are exactly the q columns/rows of A, then we will say that the columns/rows of A are linearly dependent//independent over the real (or complex) numbers.

2. **Comments on the definitions.**

   The comments on the definitions concerned with linear combinations of column/row vectors also apply here:—

   (1) For simplicity of presentation, we will focus on linear dependence/independence for column/row vectors with real entries over real numbers.
   Analogous definitions, results, and arguments for those results for linear dependence/independence for column/row vectors with complex entries over complex numbers can be obtained immediately by consistently thinking in terms of complex numbers instead of real numbers.

   (2) For further simplicity of presentation, we will state (and prove) results concerned with column vectors only, as most of the time in this course we need column vectors rather than row vectors
   The corresponding results concerned with row vectors can be obtained by 'taking transpose' consistently.

   (3) Every result concerned with linear dependence/independence of column vectors can be re-formulated in terms of matrix-vector products.

3. **Example (1). (Linear dependence and linear independence.)**

   (a) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}$.

   We verify that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent over the real numbers.
   According to definition, this amounts to verifying the statement below:—

'*For real numbers* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, *if* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \alpha_4\mathbf{u}_4 = \mathbf{0}_5$, *then* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.'

Pick any real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Suppose $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \alpha_4\mathbf{u}_4 = \mathbf{0}_5$.
Then

$$\begin{bmatrix} \alpha_1+\alpha_2+\alpha_3+\alpha_4 \\ \alpha_2+\alpha_3+\alpha_4 \\ \alpha_3+\alpha_4 \\ \alpha_4 \\ \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 3 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 4 \end{bmatrix} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \alpha_4\mathbf{u}_4 = \mathbf{0}_5.$$

By the definition of matrix equality, we obtain

$$\left\{ \begin{array}{ccccccccc} \alpha_1 & + & \alpha_2 & + & \alpha_3 & + & \alpha_4 & = & 0 \\ & & \alpha_2 & + & \alpha_3 & + & \alpha_4 & = & 0 \\ & & & & \alpha_3 & + & \alpha_4 & = & 0 \\ & & & & & & \alpha_4 & = & 0 \\ \alpha_1 & + & 2\alpha_2 & + & 3\alpha_3 & + & 4\alpha_4 & = & 0 \end{array} \right.$$

We have $\alpha_4 = 0$.
Since $\alpha_3 + \alpha_4 = 0$, we have $\alpha_3 = 0$.
Since $\alpha_2 + \alpha_3 + \alpha_4 = 0$, we have $\alpha_2 = 0$.
Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, we have $\alpha_1 = 0$.

(b) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

We verify that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ are linearly dependent over the real numbers.
According to definition, this amounts to verifying the statement below:—

'*There exist some real numbers* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ *such that* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \alpha_4\mathbf{u}_4 + \alpha_5\mathbf{u}_5 = \mathbf{0}_5$ *and* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ *are not all zero.*'

We note (with a clever observation) that

$$1\cdot\mathbf{u}_1 + 1\cdot\mathbf{u}_2 + 1\cdot\mathbf{u}_3 + 1\cdot\mathbf{u}_4 + 1\cdot\mathbf{u}_5 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}_5.$$

The numbers $1, 1, 1, 1, 1$ are certainly not all zero.

4. **Lemma (1). (Re-formulation of the notion of linear dependence in terms of homogeneous system of linear equations.)**

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are column vectors with* p *entries, and* U *is the* $(p\times q)$*-matrix given by* $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

Then the statements (LD), (LD_0) *are logically equivalent:—*

(LD) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly dependent.*
(Or equivalently: *There exist some numbers* $\alpha_1, \alpha_2, \cdots, \alpha_q$ *such that* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_q$ *are not all zero.*)

(LD_0) *The homogeneous system* $\mathcal{LS}(U, \mathbf{0}_p)$ *has some non-trivial solution.*

5. **Lemma (2). (Re-formulation of the notion of linear independence in terms of homogeneous system of linear equations.)**

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are column vectors with* p *entries, and* U *is the* $(p\times q)$*-matrix given by* $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

Then the statements (LI), (LI_0) *are logically equivalent:—*

(LI) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly independent.*
(Or equivalently: *For any numbers* $\alpha_1, \alpha_2, \cdots, \alpha_q$, *if* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_q = 0$.)

(LI_0) *The homogeneous system* $\mathcal{LS}(U, \mathbf{0}_p)$ *has no non-trivial solution.*

6. **Linear dependence and linear independence being 'opposite concept' of each other.**

   An immediate logical consequence of Lemma (1), Lemma (2) combined is the result below, which informs us that the notions of linear dependence and linear independence are 'opposite' to each other, in the sense that *being linear dependent* is the same as *being not linearly independent*, while *being linear independent* is the same as *being not linearly dependent*. This is consistent with the daily language use of the words 'dependent', 'independent'.

   Theorem (3).

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are column vectors with* p *entries.*

   Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly dependent if and only if* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are not linearly independent.*

   (And equivalently: $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are not linearly dependent if and only if* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly independent.)*

7. The argument for Lemma (1), Lemma (2) relies on the result below, labelled Lemma ($\star$), which describes a 'dictionary' between linear combinations of column vectors and matrix-vector products:—

   Lemma ($\star$).

   Let A *be an* $(p \times q)$*-matrix and* $\mathbf{t}$ *be a column vector with* q *entries.*

   Suppose that for each $j = 1, 2, \cdots, q$*, the* j*-th column of* A *is* $\mathbf{a}_j$ *and the* j*-th entry of* $\mathbf{t}$ *is* t_j*.*

   (So $A = [\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_q\ ]$ *and* $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_q \end{bmatrix}$*.)*

   Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_q\mathbf{a}_q$*.*

8. **Proof of Lemma (1).**

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are column vectors with p entries, and U is the $(p\times q)$-matrix given by $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

   (a) Suppose the statement (LD) holds: $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly dependent.*

   [*Roughwork.*
   We want to verify the statement (LD_0): '*The homogeneous system* $\mathcal{LS}(U, \mathbf{0}_p)$ *has some non-trivial solution.*'
   By definition, it suffices to name (after making an educated guess) some non-trivial solution for $\mathcal{LS}(U, \mathbf{0}_p)$, and to justify the claims involved.
   Lemma ($\star$) hints at what we may try.]

   Then, by assumption, there exist some numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ and $\alpha_1, \alpha_2, \cdots, \alpha_q$ are not all zero.

   Define $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$. By definition, $\mathbf{t} \neq \mathbf{0}_q$.

   We have $U\mathbf{t} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$.

   Then $\mathbf{t}$ is a non-trivial solution of the homogeneous system $\mathcal{LS}(U, \mathbf{0}_p)$.

   It follows that the statement (LD_0) holds.

   (b) Suppose the statement (LD_0) holds: *The homogeneous system* $\mathcal{LS}(U, \mathbf{0}_p)$ *has some non-trivial solution.*

   [*Roughwork.*
   We want to verify the statement (LD): '$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly dependent.*'
   By definition, it suffices to name (after making an educated guess) some appropriate numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ for which the equality $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$, and to justify the claims involved.
   Lemma ($\star$) hints at what we may try.]

   By assumption, we may take some non-trivial solution solution $\mathbf{t}$ for $\mathcal{LS}(U, \mathbf{0}_p)$. By definition, $U\mathbf{t} = \mathbf{0}_p$ and $\mathbf{t} \neq \mathbf{0}_q$.

   Denote by α_j the j-th entry of $\mathbf{t}$ for each $j = 1, 2, \cdots, q$. Since $\mathbf{t} \neq \mathbf{0}_q$, the numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ are not all zero.
   Also, $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = U\mathbf{t} = \mathbf{0}_p$.

   It follows that the statement (LD) holds.

9. **Proof of Lemma (2).**

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are column vectors with p entries, and U is the $(p\times q)$-matrix given by $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_q\ ]$.

(a) Suppose the statement (LI) holds: $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly independent.*

[*Roughwork.*
We want to deduce the statement (LI_0): '*The homogeneous system* $\mathcal{LS}(U,\, \mathbf{0}_p)$ *has no non-trivial solution.*'
Because the homogeneous system definitely has the trivial solution $\mathbf{0}_q$, it suffices to show that every solution of the system is the trivial solution.]

Suppose $\mathbf{t}$ is a solution of the homogeneous system $\mathcal{LS}(U,\, \mathbf{0}_p)$. Then $U\mathbf{t} = \mathbf{0}_p$.

[*Further roughwork.*
We ask whether $\mathbf{t}$ is 'forced' into being the trivial solution of $\mathcal{LS}(U,\, \mathbf{0}_p)$.]

For each j, denote the j-th entry of $\mathbf{t}$ by α_j.
We have $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = U\mathbf{t} = \mathbf{0}_p$.
Then, by assumption, $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$.
Therefore $\mathbf{t} = \mathbf{0}_q$, and $\mathbf{t}$ is the trivial solution of the homogeneous system $\mathcal{LS}(U,\, \mathbf{0}_p)$.
It follows that the statement (LI_0) holds.

(b) Suppose the statement (LI_0) holds: *The homogeneous system* $\mathcal{LS}(U,\, \mathbf{0}_p)$ *has no non-trivial solution.*

[*Roughwork.*
We want to deduce the statement (LI): '*For any numbers* $\alpha_1, \alpha_2, \cdots, \alpha_q$, *if* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_q = 0$.']

Pick any numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$. Suppose $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$.

[*Further roughwork.*
We ask whether $\alpha_1, \alpha_2, \cdots, \alpha_q$ are all zero.
Lemma ($\star$) suggests we may translate this question into something in which the assumption can be applied.]

Define $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{bmatrix}$.

We have $U\mathbf{t} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$.
Then by definition, $\mathbf{t}$ is a solution of the homogeneous system $\mathcal{LS}(U,\, \mathbf{0}_p)$.
Since $\mathcal{LS}(U,\, \mathbf{0}_p)$ has no nontrivial solution, $\mathbf{t}$ is the trivial solution of $\mathcal{LS}(U,\, \mathbf{0}_p)$. Then $\mathbf{t} = \mathbf{0}_q$.
Therefore $\alpha_1 = \alpha_2 = \cdots = \alpha_q = 0$.
It follows that the statement (LI) holds.

10. The notions of linear dependence and linear combinations are linked together in the result below:

Theorem (4). (Linear dependence re-formulated in terms of linear combinations.)

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *be column vectors with* p *entries. Then the statements below are logically equivalent:—*

($\dagger_1$) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly dependent.*

($\dagger_2$) *At least one column vector amongst* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *is a linear combination of the others.*

Remark. Theorem (4) suggests why it makes perfect sense for us to use the word 'dependence' in the phrase 'linear dependence'. In plain words, this result says that a list of finitely many (column) vectors are linearly dependent exactly when at least one of these vectors can be regarded to be 'dependent' of the others through being re-expressed as a linear combination of the others.

11. **Example (2). (Illustration of the idea in Theorem (4).)**

Write $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

We have the equality

$$(\sharp): \quad 1 \cdot \mathbf{u}_1 + (-1)\mathbf{u}_2 + 2\mathbf{u}_3 + (-2)\mathbf{u}_4 + 3\mathbf{u}_5 = \mathbf{0}_5.$$

Coincidentally, the equalities below also hold:

$$
\begin{array}{lrcl}
(\natural_1): & \mathbf{u}_1 & = & 1\cdot\mathbf{u}_2 - 2\mathbf{u}_3 + 2\mathbf{u}_4 - 3\mathbf{u}_5 \\
(\natural_2): & \mathbf{u}_2 & = & 1\cdot\mathbf{u}_1 + 2\mathbf{u}_3 + (-2)\mathbf{u}_4 + 3\mathbf{u}_5 \\
(\natural_3): & \mathbf{u}_3 & = & -\frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 + 1\cdot\mathbf{u}_4 - \frac{3}{2}\mathbf{u}_5 \\
(\natural_4): & \mathbf{u}_4 & = & \frac{1}{2}\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_2 + 1\cdot\mathbf{u}_3 + \frac{3}{2}\mathbf{u}_5 \\
(\natural_5): & \mathbf{u}_5 & = & -\frac{1}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 - \frac{2}{3}\mathbf{u}_3 + \frac{2}{3}\mathbf{u}_4
\end{array}
$$

This is in fact not just a coincidence.

Theorem (4) informs us about the logical relations of the equalities $(\sharp), (\natural_1), (\natural_2), (\natural_3), (\natural_4), (\natural_5)$:—

- Because the equality $(\sharp)$ holds, we expect at least one of the equalities of the form of $(\natural_1), (\natural_2), (\natural_3), (\natural_4), (\natural_5)$ will hold.
- Because at least one of the equalities $(\natural_1), (\natural_2), (\natural_3), (\natural_4), (\natural_5)$ holds, we expect an equality of the form of $(\sharp)$ will hold.

12. **Proof of Theorem (4).**

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column vectors with p entries.

(a) Suppose the statement $(\dagger_1)$ holds: $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent.
Then there exist some numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}_p$ and $\alpha_1, \alpha_2, \cdots, \alpha_q$ are not all zero.
By assumption, at least one of $\alpha_1, \alpha_2, \cdots, \alpha_q$ is non-zero.
Then we may assume, without loss of generality, that $\alpha_1 \neq 0$.
Now by assumption, $\mathbf{u}_1 = (-\frac{\alpha_2}{\alpha_1})\mathbf{u}_2 + (-\frac{\alpha_3}{\alpha_1})\mathbf{u}_3 + \cdots + (-\frac{\alpha_q}{\alpha_1})\mathbf{u}_q$.
Hence $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_q$.
It follows that the statement $(\dagger_2)$ holds.

(b) Suppose the statement $(\dagger_2)$ holds: at least one vector amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ is a linear combination of the others.
Then we may assume, without loss of generality, that $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_q$.
Then there exist some numbers $\beta_2, \beta_3, \cdots, \beta_q$ such that $\mathbf{u}_1 = \beta_2\mathbf{u}_2 + \beta_3\mathbf{u}_3 + \cdots + \beta_q\mathbf{u}_q$.
Therefore $1\cdot\mathbf{u}_1 + (-\beta_2)\mathbf{u}_2 + (-\beta_3)\mathbf{u}_3 + \cdots + (-\beta_q)\mathbf{u}_q = \mathbf{0}_p$.
This is a non-trivial linear relation of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly dependent.
It follows that the statement $(\dagger_1)$ holds.

13. With a purely logical consideration (known as contra-positivity), we obtain the re-formulation of Theorem (4) below.

Theorem (5). (Corollary (1) to Theorem (4).)

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are column vectors with p entries. Then the statements below are logically equivalent:—

$(\sim\dagger_1)$ *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly independent.*

$(\sim\dagger_2)$ *None of the column vector amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ is a linear combination of the others.*

Remark. The notion of linear independence can be understood through this re-formulation of Theorem (5): it corresponds to our heuristic understanding of the word *independence* in daily language.

14. Another consequence of Theorem (4) is the result below.

Theorem (6). (Corollary (2) to Theorem (4).)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ be column vectors with p entries.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly independent.

Then the statements below are logically equivalent:—

$(\dagger\dagger_1)$ *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ are linearly dependent.*

$(\dagger\dagger_2)$ *$\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.*

15. **Proof of Theorem (6).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ be column vectors with p entries.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are linearly independent.

(a) Suppose the statement ($\dagger\dagger_1$) holds: $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ are linearly dependent.
Then there exist some numbers $\alpha_1, \alpha_2, \cdots, \alpha_q, \beta$ such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q + \beta\mathbf{v} = \mathbf{0}_p$ and $\alpha_1, \alpha_2, \cdots, \alpha_q, \beta$ are not all zero.
We verify that $\beta \neq 0$, with the help of the method of proof-by-contradiction:

- Suppose it were true that $\beta = 0$.
  Then $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q = \mathbf{0}$ and $\alpha_1, \alpha_2, \cdots, \alpha_q$ were not all zero.
  Therefore $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ would be linearly dependent.
  Contradiction arises.

As $\beta \neq 0$, we have $\mathbf{v} = (-\dfrac{\alpha_1}{\beta})\mathbf{u}_1 + (-\dfrac{\alpha_2}{\beta})\mathbf{u}_2 + \cdots + (-\dfrac{\alpha_q}{\beta})\mathbf{u}_q$.

Therefore $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

It follows that the statement ($\dagger\dagger_2$) holds.

(b) Suppose the statement ($\dagger\dagger_2$) holds: $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

Then there exist some numbers $\gamma_1, \gamma_2, \cdots, \gamma_q$ such that $\mathbf{v} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_q\mathbf{u}_q$.

Therefore $\gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_q\mathbf{u}_q - 1 \cdot \mathbf{v} = \mathbf{0}_p$.

Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ are linearly dependent.

It follows that the statement ($\dagger\dagger_1$) holds.

16. With a purely logical consideration (known as contra-positivity), we obtain the re-formulation of Theorem (6) below.

Theorem (7). (Corollary (3) to Theorem (4).)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ *be column vectors with* p *entries.*

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are linearly independent.*

Then the statements below are logically equivalent:—

($\sim\dagger\dagger_1$) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q, \mathbf{v}$ *are linearly independent.*

($\sim\dagger\dagger_2$) $\mathbf{v}$ *is not a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.