1.5 Linear combinations.

0. *Assumed background.*

   - 1.1 *Matrices, matrix addition, and scalar multiplication for matrices.*
   - 1.2 *Matrix multiplication.*
   - 1.3 *Transpose, symmetry and skew-symmetry.*

   Abstract. We introduce:—

   - the notion of linear combinations.

1. **Motivation for the upcoming definition for the notion of 'linear combinations'.**

   Out of any given matrices $A, B, C, D, \cdots$ of the same size, and any given numbers $\alpha, \beta, \gamma, \delta, \cdots$, we can form, with addition and scalar multiplication alone, various new matrices of the same size

   $$\alpha A, \quad \beta B, \quad \gamma C, \quad \delta D, \quad \cdots,$$
   $$\alpha A + \beta B, \quad \alpha A + \beta B + \gamma C, \quad \alpha A + \beta B + \gamma C + \delta D, \quad \cdots$$

   This type of 'algebraic expressions' will turn up naturally in many situations (exactly because they are formed with the simplest kinds of matrix operations), and will often play fundamental roles.

   For this reason, this type of expressions deserves being given a name, and its behaviour deserves attention.

   Here, for simplicity, we only consider the special situation in which the matrices under questions are column vectors and row vectors.

2. **Definition. (Linear combination of column/row vectors over real (or complex) numbers.)**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column/row vectors with p real (or complex) entries. (These q vectors are not assumed to be pairwise distinct.)

   (1) *Any expression of the form*
   $$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q,$$
   in which $\alpha_1, \alpha_2, \cdots, \alpha_q$ are real (or complex) numbers, is called a **linear combination of the column/row vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ with respect to the real (or complex) scalars** $\alpha_1, \alpha_2, \cdots, \alpha_q$.

   (2) *Let $\mathbf{v}$ be a column/row vector with p real (or complex) entries.*
   We say $\mathbf{v}$ is a **linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ over the real (or complex) numbers** *if the statement (LC) holds:*

   (LC) *There exist some real (or complex) numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q$.*

   Remarks on terminologies.

   (a) An equality that reads
   $$\text{`}\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q\text{'}$$
   is called a **linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$**.

   (b) i. If $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are the j_1-th, j_2-th, ..., j_q-th columns/rows in a matrix, say, A, with real (or complex) entries, we will call a linear combination of these q column/row vectors a **linear combination of the j_1-th, j_2-th, ..., j_q-th columns/rows of** A.

   ii. Furthermore, if A is a $(p \times q)$-matrix and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ are exactly the q columns/rows of A, such a linear combination is called a **linear combination of the columns/rows of** A.

3. **Comment (1) on the definitions.**

   From now on, for simplicity of presentation, we will focus on linear combinations of column/row vectors here with real entries over real numbers:—

   - The phrase '*with p entries*' will read '*with p real entries*'.
   - The phrase '*the numbers/scalars $\alpha_1, \alpha_2, \cdots$*' will read '*the real numbers/scalars $\alpha_1, \alpha_2, \cdots$*'.
   - The phrase '*a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots$*' will read '*a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots$ over the real numbers*'.
   - The phrase '*the matrix A*' will read '*the matrix A with real entries*'.

Et cetera.

Our conceptual understanding will not be weakened. When we consistently change the reference to real numbers to complex numbers in the definitions, theorems, and proofs stated here, we will recover the corresponding definitions, theorems and proofs concerned with linear combinations of column/row vectors with complex entries over complex numbers.

This is because the definitions, theorems, and proofs we state here regarding the notion of linear combinations rely only on the the basic 'algebraic properties' of matrix addition, scalar multiplication, and matrix multiplication, which in turn rely only on the basic 'algebraic properties' of addition, subtraction, multiplication and division for real numbers. But these basic 'algebraic properties' of addition, subtraction, multiplication and division for complex numbers are formally the 'same' as that for real numbers.

4. **Lemma (1). ('Dictionary' between linear combinations of column vectors and that of row vectors.)**

Suppose $\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *are column/row vectors with* p *entries, and* $\alpha_1, \alpha_2, \cdots, \alpha_q$ *are scalars.*

Then the statements below are logically equivalent:

(1) *The column/row vector* $\mathbf{v}$ *is a linear combination of the column/row vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *with respect to scalars* $\alpha_1, \alpha_2, \cdots, \alpha_q$.

(2) *The row/column vector* $\mathbf{v}^t$ *is a linear combination of the row/column vectors* $\mathbf{u}_1^t, \mathbf{u}_2^t, \cdots, \mathbf{u}_q^t$ *with respect to scalars* $\alpha_1, \alpha_2, \cdots, \alpha_q$.

Proof of Lemma (1). Exercise.

5. **Comment (2) on the definitions, as a remark on Lemma (1).**

Because of Lemma (1), for each result involving the notion of linear combinations of column vectors alone, we can immediately obtain an analogous results concerned with row vectors by changing the word 'column' to 'row', and taking 'transpose' for all matrices and vectors involved in the statement and its proof.

So, from now on, for further simplicity of presentation, we will state (and prove) results concerned with column vectors only, as most of the time in this course we need column vectors rather than row vectors.

6. **Lemma (2). ('Dictionary' between linear combinations and matrix-vector products.)**

Let A *be an* $(p \times q)$*-matrix and* $\mathbf{t}$ *be a column vector with* q *entries.*

Suppose that for each $j = 1, 2, \cdots, q$, *the* j*-th column of* A *is* $\mathbf{a}_j$ *and the* j*-th entry of* $\mathbf{t}$ *is* t_j.

(So $A = [\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_q\ ]$ *and* $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_q \end{bmatrix}$.*)*

Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_q\mathbf{a}_q$.

7. **Comment (3) on the definitions, as a remark on Lemma (2).**

Lemma (2) links up the definitions (on linear combinations of column vectors) with matrix multiplication.

In plain words, the result says:—

- *A given column vector is a linear combination of the columns of a given matrix if and only if the column vector concerned is resultant from multiplying the matrix concerned from the left to some appropriate column vector.*

While the result looks innocent, it will serve as a useful tool, because sometimes it is more convenient to think of a linear combination of column vectors as a matrix-vector product, and some other times it is more convenient to think of a matrix-vector product as a linear combination of the columns of the matrix involved in the product.

8. **Proof of Lemma (2).**

Let A be an $(p \times q)$-matrix and $\mathbf{t}$ be a column vector with q entries.

Suppose that for each $j = 1, 2, \cdots, q$, the j-th column of A is $\mathbf{a}_j$ and the j-th entry of $\mathbf{t}$ is t_j.

For each i, j, we denote the (i, j)-th entry of A by a_{ij}.

- The i-th entry of $A\mathbf{t}$ is given by $\displaystyle\sum_{j=1}^{n} a_{ij}t_j = t_1a_{i1} + t_2a_{i2} + \cdots + t_qa_{iq}$.
- For each j, the i-th entry of $\mathbf{a}_j$ (which is the j-th column of A) is a_{ij}.
  Then the i-th entry of $t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_q\mathbf{a}_q$ is $t_1a_{i1} + t_2a_{i2} + \cdots + t_qa_{iq}$.

The corresponding entries of $A\mathbf{t}$, $t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_q\mathbf{a}_q$ agree with each other.

Hence $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_q\mathbf{a}_q$ indeed.

9. **Example (1). (Simple concrete examples about linear combinations of column/row vectors.)**

(a) $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = 1\cdot\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}.$

So $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ is the linear combination of $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ with respect to the scalars $1, 2, 3, 4, 5$.

A manifestation of the same relation is the equality below:

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \left[\begin{array}{c|c|c|c|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{array}\right]\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}.$$

(b) $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = 1\cdot\begin{bmatrix}1\\3\\5\\7\\9\end{bmatrix} + (-1)\begin{bmatrix}0\\2\\4\\6\\8\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\\3\\5\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\0\\3\\6\end{bmatrix} + 0\cdot\begin{bmatrix}0\\0\\0\\0\\2\end{bmatrix}.$

So $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ is also a linear combination of $\begin{bmatrix}1\\3\\5\\7\\9\end{bmatrix}, \begin{bmatrix}0\\2\\4\\6\\8\end{bmatrix}, \begin{bmatrix}0\\0\\1\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\2\end{bmatrix}$ with respect to the scalars $1, -1, 2, -1, 0$.

A manifestation of the same relation is the equality below:

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \left[\begin{array}{c|c|c|c|c}1&0&0&0&0\\3&2&0&0&0\\5&4&1&0&0\\7&6&3&3&0\\9&8&5&6&2\end{array}\right]\begin{bmatrix}1\\-1\\2\\-1\\0\end{bmatrix}.$$

(c) $\begin{bmatrix}-4 & 5 & 4 & -1\end{bmatrix} = 2\begin{bmatrix}1 & 2 & 1 & 0\end{bmatrix} + 3\begin{bmatrix}2 & 3 & 2 & 1\end{bmatrix} - 4\begin{bmatrix}3 & 2 & 1 & 1\end{bmatrix}.$

So $\begin{bmatrix}-4 & 5 & 4 & -1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1 & 2 & 1 & 0\end{bmatrix}, \begin{bmatrix}2 & 3 & 2 & 1\end{bmatrix}, \begin{bmatrix}3 & 2 & 1 & 1\end{bmatrix}$ with respect to the scalar $2, 3, -4$.

A manifestation of the same relation is the equality below:

$$\begin{bmatrix}-4 & 5 & 4 & -1\end{bmatrix} = \begin{bmatrix}2 & 3 & -4\end{bmatrix}\left[\begin{array}{cccc}1&2&1&0\\\hline 2&3&2&1\\\hline 3&2&1&1\end{array}\right].$$

With transpose being taken on both sides of the equality above, we obtain an equality which gives the same information:

$$\begin{bmatrix}-4\\5\\4\\-1\end{bmatrix} = \left[\begin{array}{c|c|c}1&2&3\\2&3&1\\1&2&1\\0&1&1\end{array}\right]\begin{bmatrix}2\\3\\-4\end{bmatrix}.$$

But this last equality is just giving the same information as this equality about linear combinations of column vectors.

$$\begin{bmatrix}-4\\5\\4\\-1\end{bmatrix} = 2\begin{bmatrix}1\\2\\1\\0\end{bmatrix} + 3\begin{bmatrix}2\\3\\2\\1\end{bmatrix} - 4\begin{bmatrix}3\\2\\1\\1\end{bmatrix}.$$

10. **Theorem (3). (Addition and scalar multiplication of linear combinations.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column vectors with p entries.

The statements below are true:

(a) *The zero vector $\mathbf{0}_p$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.*

(b) *The sum of any two linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.*

(c) *Every scalar multiple of any linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

11. **Proof of Theorem (3).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column vectors with p entries.

(a) [Ask: Can we name some appropriate numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ for which the equality $\mathbf{0}_p = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_q$ holds?]

We have $\mathbf{0} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + \cdots + 0 \cdot \mathbf{u}_q$.

Then by definition, $\mathbf{0}_p$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

(b) Suppose $\mathbf{v}, \mathbf{w}$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

Then, by definition, there exist some numbers $\beta_1, \beta_2, \cdots, \beta_q$ such that $\mathbf{v} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_q\mathbf{u}_q$.

Also, there exist some numbers $\gamma_1, \gamma_2, \cdots, \gamma_q$ such that $\mathbf{w} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_q\mathbf{u}_q$.

[Ask: Can we name some appropriate numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ for which the equality $\mathbf{v} + \mathbf{w} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_q\mathbf{u}_q$ holds?]

Note that $\mathbf{v} + \mathbf{w} = ... = (\beta_1 + \gamma_1)\mathbf{u}_1 + (\beta_2 + \gamma_2)\mathbf{u}_2 + \cdots + (\beta_q + \gamma_q)\mathbf{u}_q$, and $\beta_1 + \gamma_1, \beta_2 + \gamma_2, \cdots, \beta_q + \gamma_q$ are well-defined as numbers.

Then by definition, $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

(c) Exercise.

12. **Theorem (4). (Linear combination of linear combinations.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *be column vectors with* p *entries.*

Every linear combination of (finitely many) linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

Remark. In fact, Theorem (4) is saying the same thing as Statement (b) and Statement (c) in Theorem (3) combined.

Its conclusion part can be formulated as:

> *For any column vector* $\mathbf{x}$ *with* p *entries, if* $\mathbf{x}$ *is a linear combination of some column vectors* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ *with* p *entries, which are themselves linear combinations of column vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *with* p *entries, then* $\mathbf{x}$ *itself is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

13. **Proof of Theorem (4).**

[This argument carries the same essence of the argument for Statement (b) and Statement (c) in Theorem (3).]

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column vectors with p entries.

Pick any column vector $\mathbf{x}$ with p entries.

Suppose $\mathbf{x}$ is a linear combination of (finitely many) column vectors with p entires, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, which are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

[Reminder: We want to see why $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.]

By definition, $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.

Then there exist some numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$.

[Ask: Can we link up the $\mathbf{u}_j$'s with the $\mathbf{v}_i$'s so as to see that $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$?]

By assumption, for each $j = 1, 2, \cdots, q$, there exist some numbers $\beta_{1j}, \beta_{2j}, \cdots, \beta_{qj}$ such that $\mathbf{v}_j = \beta_{1j}\mathbf{u}_1 + \beta_{2j}\mathbf{u}_2 + \cdots + \beta_{qj}\mathbf{u}_q$.

Then

$$\begin{aligned}
\mathbf{x} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n \\
&= \alpha_1(\beta_{11}\mathbf{u}_1 + \beta_{21}\mathbf{u}_2 + \cdots + \beta_{q1}\mathbf{u}_q) + \alpha_2(\beta_{12}\mathbf{u}_1 + \beta_{22}\mathbf{u}_2 + \cdots + \beta_{q2}\mathbf{u}_q) \\
&\quad + \cdots + \alpha_n(\beta_{1p}\mathbf{u}_1 + \beta_{2p}\mathbf{u}_2 + \cdots + \beta_{qn}\mathbf{u}_q) \\
&= (\beta_{11}\alpha_1 + \beta_{12}\alpha_2 + \cdots + \beta_{1n}\alpha_n)\mathbf{u}_1 + (\beta_{21}\alpha_1 + \beta_{22}\alpha_2 + \cdots + \beta_{2n}\alpha_n)\mathbf{u}_2 \\
&\quad + \cdots + (\beta_{n1}\alpha_1 + \beta_{n2}\alpha_2 + \cdots + \beta_{qn}\alpha_n)\mathbf{u}_q
\end{aligned}$$

Note that $(\beta_{k1}\alpha_1 + \beta_{k2}\alpha_2 + \cdots + \beta_{kn}\alpha_n)$ is well-defined as a number for each $k = 1, 2, \cdots, q$.

Then $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

Remark. We shall later also give an alternative argument for Theorem (), with an application of mathematical induction.

14. **Example (2). (Concrete illustration of the content of Theorem (3), Theorem (4), and the 'algebraic content' of their proofs.)**

    Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2, \mathbf{x}$ be column vectors with 5 entries.

    Suppose $\mathbf{v}_1 = 2\mathbf{u}_1 - 3\mathbf{u}_2 + 4\mathbf{u}_3 - \mathbf{u}_4$, $\mathbf{v}_2 = \mathbf{u}_1 + 2\mathbf{u}_2 - 3\mathbf{u}_3 + 4\mathbf{u}_4$, $\mathbf{v}_3 = -\mathbf{u}_1 + 3\mathbf{u}_2 + 2\mathbf{v}_3 + \mathbf{u}_4$.

    Note that each of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

    (a) Note that $\mathbf{0}_5 = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3 + 0 \cdot \mathbf{u}_4$.
    Hence $\mathbf{0}_5$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

    (b) Suppose $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$. (So $\mathbf{w}_1$ is the sum of $\mathbf{v}_1, \mathbf{v}_2$.)
    Note that $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 = (2\mathbf{u}_1 - 3\mathbf{u}_2 + 4\mathbf{u}_3 - \mathbf{u}_4) + (\mathbf{u}_1 + 2\mathbf{u}_2 - 3\mathbf{u}_3 + 4\mathbf{u}_4) = 3\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3 + 3\mathbf{u}_4$.
    Hence $\mathbf{w}_1$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

    (c) Suppose $\mathbf{w}_2 = 4\mathbf{v}_3$. (So $\mathbf{w}_2$ is a scalar multiple of $\mathbf{v}_3$.)
    Note that $4\mathbf{w}_2 = 4(-\mathbf{u}_1 + 3\mathbf{u}_2 + 2\mathbf{v}_3 + \mathbf{u}_4) = -4\mathbf{u}_1 + 12\mathbf{u}_2 + 8\mathbf{v}_3 + 4\mathbf{u}_4$.
    Hence $\mathbf{w}_2$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

    (d) Suppose $\mathbf{w}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 - 4\mathbf{v}_3$. (So $\mathbf{w}_3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.)
    Note that

$$\begin{aligned} \mathbf{w}_3 &= 3\mathbf{v}_1 + 2\mathbf{v}_2 - 4\mathbf{v}_3 \\ &= 3(2\mathbf{u}_1 - 3\mathbf{u}_2 + 4\mathbf{u}_3 - \mathbf{u}_4) + 2(\mathbf{u}_1 + 2\mathbf{u}_2 - 3\mathbf{u}_3 + 4\mathbf{u}_4) - 4(-\mathbf{u}_1 + 3\mathbf{u}_2 + 2\mathbf{v}_3 + \mathbf{u}_4) \\ &= 12\mathbf{u}_1 - 17\mathbf{u}_2 - 2\mathbf{u}_3 - \mathbf{u}_4 \end{aligned}$$

    Hence $\mathbf{w}_3$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

15. **Example (3). (Baby examples on checking whether a given vector is a linear combination of a number of given vectors.)**

    The illustrations below suggest that such problems may become non-trivial when the vectors concerned have three or more entries. The key to the problem seems to be related to mathematical objects that we call *systems of linear equations.*

    (a) Let $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
    We want to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, and to find a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2$ if such exists.

    i. *Question. Suppose* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2$*. Then how is* $\mathbf{v}$ *related with* $\mathbf{u}_1, \mathbf{u}_2$ *through a linear relation?*
    Answer. If $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, then by definition, there are some numbers α_1, α_2 such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$.

    ii. *Further question.* But what are α_1, α_2?
    Answer. We have the equalities

$$\begin{bmatrix} \alpha_1 & + & \alpha_2 \\ \alpha_1 & - & \alpha_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 = \mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

    The definition of matrix equality gives

$$\begin{cases} \alpha_1 + \alpha_2 = 4 \\ \alpha_1 - \alpha_2 = 2 \end{cases}$$

    Then $2\alpha_1 = (\alpha_1 + \alpha_2) + (\alpha_1 - \alpha_2) = 4 + 2 = 6$. Therefore $\alpha_1 = 3$.
    Also $2\alpha_2 = (\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) = 4 - 2 = 2$. Therefore $\alpha_1 = 1$.

    iii. Up to now, we have discovered that:—
    - *If* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2$ *then it is necessary for the equality* $\mathbf{v} = 3\mathbf{u}_1 + \mathbf{u}_2$ *to hold.*

    iv. *Question.* But is it indeed true that $\mathbf{v} = 3\mathbf{u}_1 + \mathbf{u}_2$?
    Answer. By direct computing, we see that

$$3\mathbf{u}_1 + 1\mathbf{u}_2 = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \mathbf{v}.$$

    Hence $\mathbf{v}$ is indeed a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, through the linear relation $\mathbf{v} = 3\mathbf{u}_1 + \mathbf{u}_2$.

(b) Let $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and to find a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ if such exists.

i. *Question.* *Suppose* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. *Then how is* $\mathbf{v}$ *related with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *through a linear relation?*
Answer. If $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, then by definition, there are some numbers $\alpha_1, \alpha_2, \alpha_3$ such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3$.

ii. *Further question.* But what are $\alpha_1, \alpha_2, \alpha_3$?
Answer. We have the equalities

$$\begin{bmatrix} \alpha_1 & + & \alpha_2 & + & \alpha_3 \\ & & \alpha_2 & + & \alpha_3 \\ & & & & \alpha_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 = \mathbf{v} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}.$$

The definition of matrix equality gives

$$\left\{ \begin{array}{ccccccc} \alpha_1 & + & \alpha_2 & + & \alpha_3 & = & 5 \\ & & \alpha_2 & + & \alpha_3 & = & 3 \\ & & & & \alpha_3 & = & 1 \end{array} \right.$$

Note that $\alpha_3 = 1$.
Then $\alpha_2 = (\alpha_2 + \alpha_3) - \alpha_2 = 3 - 1 = 2$.
Now $\alpha_1 = (\alpha_1 + \alpha_2 + \alpha_3) - (\alpha_1 + \alpha_2) = 5 - 3 = 2$.

iii. Up to now, we have discovered that:—
- *If* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *then it is necessary for the equality* $\mathbf{v} = 2\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3$ *to hold.*

iv. *Question.* But is it indeed true that $\mathbf{v} = 2\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3$?
Answer. By direct computing, we see that

$$2\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3 = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \mathbf{v}.$$

Hence $\mathbf{v}$ is indeed a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, through the linear relation $\mathbf{v} = 2\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3$.

(c) Let $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and to find a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ if such exists.

i. *Question.* *Suppose* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. *Then how is* $\mathbf{v}$ *related with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *through a linear relation?*
Answer. If $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, then by definition, there are some numbers $\alpha_1, \alpha_2, \alpha_3$ such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3$.

ii. *Further question.* But what are $\alpha_1, \alpha_2, \alpha_3$?
Answer. We have the equalities

$$\begin{bmatrix} \alpha_1 & + & 2\alpha_2 & + & 3\alpha_3 \\ \alpha_1 & + & \alpha_2 & + & 2\alpha_3 \\ & & \alpha_2 & + & \alpha_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 = \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

The definition of matrix equality gives

$$\left\{ \begin{array}{ccccccc} \alpha_1 & + & 2\alpha_2 & + & 3\alpha_3 & = & 2 \\ \alpha_1 & + & \alpha_2 & + & 2\alpha_3 & = & 1 \\ & & \alpha_2 & + & \alpha_3 & = & 0 \end{array} \right.$$

Note that $\alpha_2 + \alpha_3 = 0$.
Also note that $\alpha_2 + \alpha_3 = (\alpha_1 + 2\alpha_2 + 3\alpha_3) - (\alpha_1 + \alpha_2 + 2\alpha_2) = 2 - 1 = 1$.
Then $0 = \alpha_2 + \alpha_3 = 1$, which is impossible.
Hence $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(d) Let $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, and to find a linear relation relating $\mathbf{v}$ with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ if such exists.

i. *Question.* *Suppose* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$. *Then how is* $\mathbf{v}$ *related with* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *through a linear relation?*
Answer. If $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, then by definition, there are some numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \alpha_4\mathbf{u}_4$.

ii. *Further question.* But what are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$?
Answer. We have the equalities

$$\begin{bmatrix} \alpha_1 & + & \alpha_2 & + & \alpha_3 & + & \alpha_4 \\ & & \alpha_2 & + & \alpha_3 & + & 2\alpha_4 \\ & & & & \alpha_3 & + & 3\alpha_4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \alpha_4\mathbf{u}_4 = \mathbf{v} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

The definition of matrix equality gives

$$\left\{ \begin{array}{ccccccccc} \alpha_1 & + & \alpha_2 & + & \alpha_3 & + & \alpha_4 & = & 6 \\ & & \alpha_2 & + & \alpha_3 & + & 2\alpha_4 & = & 4 \\ & & & & \alpha_3 & + & 3\alpha_4 & = & 2 \end{array} \right.$$

By observation, these equalities in turn give relations which relate each of $\alpha_1, \alpha_2, \alpha_3$ in terms of α_4 alone:

- We have $\alpha_1 - \alpha_4 = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - (\alpha_2 + \alpha_3 + 2\alpha_4) = 6 - 4 = 2$. Then $\alpha_1 = 2 + \alpha_4$.
- We have $\alpha_2 - \alpha_4 = (\alpha_2 + \alpha_3 + 2\alpha_4) - (\alpha_3 + 3\alpha_4) = 4 - 2 = 2$. Then $\alpha_2 = 2 + \alpha_4$
- We have $\alpha_3 - \alpha_4 = 2$. Then $\alpha_3 = 2 + \alpha_4$.

iii. Up to now, we have discovered that:—

- *If* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *then it is necessary for the equality*

$$\mathbf{v} = (2+t)\mathbf{u}_1 + (2+t)\mathbf{u}_2 + (2+t)\mathbf{u}_3 + t\mathbf{t}_4$$

to hold for some number t.

iv. *Question.* But is it indeed true that $\mathbf{v} = (2+t)\mathbf{u}_1 + (2+t)\mathbf{u}_2 + (2+t)\mathbf{u}_3$ for some number t?
Answer. It happens that when, say, $t = 0$, we have

$$(2+t)\mathbf{u}_1 + (2+t)\mathbf{u}_2 + (2+t)\mathbf{u}_3 + t\mathbf{u}_4 = 2\mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3 + 0 \cdot \mathbf{u}_4 = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \mathbf{v}.$$

Hence $\mathbf{v}$ is indeed a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, through the linear relation $\mathbf{v} = 2\mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3 + 0 \cdot \mathbf{u}_4$.

Remark. As seen in this example, determining whether a given vector is a linear combination of a number of given vectors looks difficult, because the process involves solving a complicated system of 'simultaneous equations' which arises naturally in the process.

16. **Alternative argument for Theorem (4), as an illustration on the application of mathematical induction.**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ be column vectors with p entries. (They are kept fixed throughout the rest of the argument.)

By applying mathematical induction, and by consciously applying Theorem (3), we verify the statement

> '*For any positive integer* s, *if* $\alpha_1, \alpha_2, \cdots, \alpha_s$ *are numbers, and* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s$ *are linear combinations of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$, *and then* $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_s\mathbf{v}_s$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.'

Denote by $P(s)$ the proposition below:

> 'If $\alpha_1, \alpha_2, \cdots, \alpha_s$ are numbers, and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ then $\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v}_2 + \cdots + \alpha_s\mathbf{v}_s$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.'

We verify $P(1)$:

> Suppose α_1 is a number, and $\mathbf{v}_1$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
> Then by Theorem (3), $\alpha_1\mathbf{v}_1$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.

Suppose $P(k)$ is true.

Note that $P(k+1)$ reads:

> 'If and $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_{k+1}$ are numbers, and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$, then $\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{k+1}\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.'

With the help of $P(k)$, we verify $P(k+1)$:

Suppose $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_{k+1}$ are numbers and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
By $P(k)$, $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
By $P(1)$, $\alpha_{k+1}\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
Then, by Theorem (3), $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \alpha_{k+1}\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
Therefore $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(s)$ is true for any positive integer s.