1.4 Commuting matrices versus non-commuting matrices.

0. *Assumed background.*

   - 1.1 *Matrices, matrix addition, and scalar multiplication for matrices.*
   - 1.2 *Matrix multiplication.*

   Abstract. We introduce:—

   - the notion of commuting matrices,
   - the non-commutativity of matrix multiplication in general.

   We also note the ideas in:—

   - the method of proof by mathematical induction,
   - the notion of counter-examples for refuting a statement.

   More on the ideas will be covered in the *appendices*.

1. **Definition. (Commuting matrices.)**

   Suppose A, B are matrices.

   Then A, B are said to **commute with each other** *if and only if the equality $AB = BA$ holds.*

   We can also say that A, B are a pair of commuting matrices.

   Remark. Commutativity is guaranteed for matrix addition. There is a 'Law of Commutativity for matrix addition', which reads: '*Suppose A, B are matrices of the same size. Then $A + B = B + A$*'.

   But it is the question of commutativity is highly nontrivial for matrix multiplication.

   Given an arbitrary pair of matrices, there is no guarantee that they commute with each other, as the examples below illustrate. This is exactly why such a definition is useful.

2. **Example (1). (Examples and non-examples of commuting matrices.)**

   (a) The $(n \times n)$-zero matrix commute with every $(n \times n)$-square matrix.

   (b) The $(n \times n)$-identity matrix commute with every $(n \times n)$-square matrix.

   (c) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$.

   ⇐ this holds for arbitrary pairs of diagonal matrices of same size.

   $\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

   We have

   $$AB = \cdots = \begin{bmatrix} 12 & 0 \\ 0 & 15 \end{bmatrix}, \quad BA = \cdots = \begin{bmatrix} 12 & 0 \\ 0 & 15 \end{bmatrix}.$$

   Then $AB = BA$. Therefore A, B commute with each other.

   (d) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

   We have

   $$AB = \cdots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad BA = \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   $AB = BA = I_3$

   " $B = A^{-1}$, it's the inverse of A "

   $A \cdot A^{-1} = A^{-1} \cdot A = I_n$

   Then $AB = BA$. Therefore A, B commute with each other.

   (e) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$.

   $$AB = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix}.$$

   $$BA = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \cdots = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 7 & 2 \\ 2 & 4 & 1 \end{bmatrix}.$$

   So it follows that $AB \neq BA$.

   (AB, BA are not even comparable, as they are not of the same size.)

(f) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \cdots = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathcal{O}_{2\times 2}.$$

x, y are 2 numbers, if $x \neq 0$, $y \neq 0$ $\Rightarrow xy \neq 0$

$B \neq I_2$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \cdots = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A.$$

So it follows that $AB \neq BA$. Therefore A, B do not commute with each other.

(g) Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

We have

$$AB = \cdots = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad BA = \cdots = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $AB \neq BA$. Therefore A, B do not commute with each other.

Question: Diagonal matrix: $\Lambda := \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$\lambda_i \neq \lambda_j$ for $1 \leq i, j \leq n$ if $i \neq j$

Find all matrices A such that $A\Lambda = \Lambda \cdot A$

(h) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

We have

$$A : AB = \cdots = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad BA = \cdots = \overset{=B}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}.$$

Then $AB \neq BA$. Therefore A, B do not commute with each other.

counter-example

Remark. What some parts of Example (1) has told us is that the statements $(\star), (\star\star)$ are false:—

($\star$) *Suppose A is an $(m \times n)$-matrix and B is an $(n \times m)$-matrix. Then $AB = BA$.*

($\star\star$) *Suppose A, B are $(n \times n)$-matrices. Then $AB = BA$.*

The content of part (e) of Example (1) is referred to as a counter-example against the statement ($\star$).

The content of part (f) of Example (1) is referred to as a $\boxed{\text{counter-example}}$ against the statement ($\star\star$).

Further remark. Were either of ($\star$), ($\star\star$) true, we would have something that would be called 'Law of Commutativity for matrix multiplication'. The falsity of both ($\star$), ($\star\star$) says that there is no such thing as 'Law of Commutativity for matrix multiplication'.

Assumption   conclusion

3. **Lemma (1).**

Let A, B be $(p \times p)$-square matrix. $\boxed{\text{Suppose } A, B \text{ commute with each other.}}$ Then, for any positive integer n, the matrices A, B^n commute with each other.

Proof of Lemma (1).

Let A, B be $(p \times p)$-square matrix. Suppose A, B commute with each other.

[We want to deduce

$$\text{‘}\textit{for any positive integer } n, \underbrace{\textit{the matrices } A, B^n \textit{ commute with each other}}_{P(n)}\text{’}$$

with an application of mathematical induction:—

- We deduce $\underbrace{\textit{the matrices } A, B^1 \textit{ commute with each other.}}_{P(1)}$

- We deduce that

$$\textit{for each positive integer } k, \textit{ if } \underbrace{\textit{the matrices } A, B^k \textit{ commute with each other}}_{P(k)}$$
$$\textit{then } \underbrace{\textit{the matrices } A, B^{k+1} \textit{ commute with each other}}_{P(k+1)}.$$

We complete the procedure by citing the 'Principle of Mathematical Induction'.]

We denote by $P(n)$ the proposition '*A, B^n commute with each other*'.

- Note that $B^1 = B$.
  By assumption, A, B commute with each other.
  Hence $P(1)$ is true.

- Let k be a positive integer. Suppose $P(k)$ is true. (Then $AB^k = B^kA$.)
  [We want to verify the equality $AB^{k+1} = B^{k+1}A$.]
  We have

$$\begin{aligned} AB^{k+1} &= (AB)B^k \quad \text{(by associativity of matrix multiplication)} \\ &= (BA)B^k \quad \text{(by } P(1)\text{)} \\ &= B(AB^k) \quad \text{(by associativity of matrix multiplication)} \\ &= B(B^kA) \quad \text{(by } P(k)\text{)} \\ &= B^{k+1}A \quad \text{(by associativity of matrix multiplication)} \end{aligned}$$

  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n.

4. **Lemma (2).**

   Let A, B be $(p \times p)$-square matrix. Suppose A, B commute with each other. Then, for any positive integers m, n, the matrices A^m, B^n commute with each other.

   Proof of Lemma (2). Exercise. (Apply Lemma (1).)

5. **Theorem (3). (Binomial Theorem for commuting matrices.)**

   Let A, B be $(p \times p)$-square matrix. Suppose A, B commute with each other. Then, for any positive integer n, the equality

$$(A+B)^n = A^n + \binom{n}{1}A^{n-1}B + \binom{n}{2}A^{n-2}B^2 + \cdots + \binom{n}{k}A^{n-k}B^k + \cdots + \binom{n}{n-1}AB^{n-1} + B^n$$

   holds .

   Proof of Theorem (3). Exercise. (Apply mathematical induction.)

   Analogue with real numbers $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

6. **Comment on Theorem (3).** When we delete the words

   '*Suppose A, B commute with each other.*'

   from the statement of Theorem (3), we will end up with the *false* statement $(\star\star\star)$:—

   $(\star\star\star)$ *Suppose A, B are $(p \times p)$-square matrix. Then, for any positive integer n, the equality*

$$(A+B)^n = A^n + \binom{n}{1}A^{n-1}B + \binom{n}{2}A^{n-2}B^2 + \cdots + \binom{n}{k}A^{n-k}B^k + \cdots + \binom{n}{n-1}AB^{n-1} + B^n$$

   holds .

   To convince ourselves that $(\star\star\star)$ is indeed false, we present a counter-example against it (obtained by re-using the work in part (f) of Example (1)):

   Why we can prove it in general, take the following as an example:

   - Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. (Take $n = 2$).

     We have $(A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

     We also have $A^2 = AB = \mathcal{O}_{2\times 2}$, and $B^2 = B$.

     Then $A^2 + \binom{2}{1}AB + B^2 = A^2 + 2AB + B^2 = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

     Therefore $(A+B)^2 \neq A^2 + \binom{2}{1}AB + B^2$.

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = A^2 + AB + BA + B^2$$

$$\Rightarrow (A+B)^2 = A^2 + 2AB + B^2. \Rightarrow AB = BA.$$

$$\Rightarrow AB + BA = 2AB$$