Transpose, symmetry and skew-symmetry.

- 0. Assumed background.
 - 1.1 Matrices, matrix addition, and scalar multiplication for matrices.
 - 1.2 Matrix multiplication.

Abstract. We introduce:—

- the notion of transpose,
- the notions of symmetry and skew-symmetry.

In the appendix, we digress onto the notion of definition, theorem, proof, and the format which dictates how they are to be read.

1. Definition. (Transpose of a matrix.)

Let A be an $(m \times n)$ -matrix, whose (i, j)-th entry is denoted by a_{ii} .

The **transpose of** *A* is the $(n \times m)$ -matrix whose (k, ℓ) -th entry is given by $a_{\ell k}$ It is denoted by A^t .

Remark. In symbolic terms, what this definition says is:—

$$If A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} then A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

2. Example (1). (Transpose of a matrix.)

Suppose
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$
, $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}$.

Then
$$A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$
, $B^t = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$ and $C^t = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

of transpose with

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addition (a) Note that
$$A + B = \begin{bmatrix} 2 & 5 & 3 \\ 2 & 2 & 3 \end{bmatrix}$$
. Then $(A + B)^t = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$.

We have
$$A^t + B^t = \dots = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$$
. So $(A+B)^t = A^t + B^t$ (in this example).

(b) Note that
$$AC = \cdots = \begin{bmatrix} 4 & 13 \\ 2 & 7 \end{bmatrix}$$
. Then $(AC)^t = \begin{bmatrix} 4 & 2 \\ 13 & 7 \end{bmatrix}$
We have $C^tA^t = \cdots = \begin{bmatrix} 4 & 2 \\ 13 & 7 \end{bmatrix}$. So $(AC)^t = C^tA^t$ (in this example).

3. Theorem (1). (Basic properties of transpose.)

Suppose A, B are $(m \times n)$ -matrices, C is an $(n \times p)$ -matrix, and λ is a number. Then:—

(1) Suppose A is an
$$(m \times n)$$
-matrix. Then $(A^t)^t = A$.

- (2) Suppose A, B are $(m \times n)$ -matrices. Then $(A + B)^t = A^t + B^t$.
- (3) Suppose *A* is an $(m \times n)$ -matrix, and λ is a number. Then $(\lambda A)^t = \lambda A^t$.
- (4) Suppose A is an $(m \times n)$ -matrices, and C is an $(n \times p)$ -matrix. Then $(AC)^t = C^t A^t$.

Proof of Statement (4) of Theorem (1).

Suppose A is a $(m \times n)$ -matrix, and C is an $(n \times p)$ -matrix. (So AC is an $(m \times p)$ -matrix, and $(AC)^t$ is a $(p \times m)$ -matrix.) (By definition, A^t is an $(n \times m)$ -matrix, and C^t is a $(p \times n)$ -matrix. So $C^t A^t$ is well-defined as a $(p \times m)$ -matrix.) Denote the (i, j)-th entry of A by a_{ij} . Denote the (k, ℓ) -th entry of C by $c_{k\ell}$.

Fix any $\ell = 1, 2, \dots, p$ and $i = 1, 2, \dots, m$.

first thoughts on the equality

(AC) = CtAt; A. (m,n) (ACT: (pxm) - machix

> ct At . (pxm)-hostix (Not a proof (1)

• By the definition of matrix multiplication, the (i, ℓ) -th entry of AC is given by $\sum a_{ij}c_{j\ell}$.

Then, by the definition of transpose, the (ℓ, i) -th entry of (AC) is given by $\sum_{i=1}^{n} a_{ij}c_{j\ell}$.

• By the definition of transpose, for each $j=1,2,\cdots,n$, the (ℓ,j) -th entry of C^t is $c_{j\ell}$, and the (j,i)-th entry of A^t is a_{ij} .

Then, by the definition of matrix multiplication, the (ℓ, i) -th entry of $C^t A^t$ is given by $\sum_{i=1}^{n} a_{ij} c_{j\ell}$.

Hence $(AC)^t = C^t A^t$.

Proof of Statements (1), (2), (3) of Theorem (1). Exercise. (Imitate what is done above.)

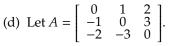
4. Definition. (Symmetric matrix and skew-symmetric matrix.)

Suppose A is an $(n \times n)$ -square matrix. Then:—

- (1) A is said to be **symmetric** if and only if $A^t = A$.
- (2) A is said to be **skew-symmetric** if and only if $A^t = -A$.
- 5. Example (2). (Examples and non-examples on symmetric matrices and skew-symmetric matrices.)
 - (a) The $(n \times n)$ -zero matrix is a symmetric matrix. It is also a skew-symmetric matrix.
 - (b) The identity matrix is a symmetric matrix. It is not skew-symmetric.
 - (c) Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 6 \end{bmatrix}$.

Note that $A^t = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 6 \end{bmatrix} = A$. Then A is symmetric.

Note that $A^t \neq -A$. Then A is not skew-symmetric.



i.e. $A^{t=A}$ Note that $A^t = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -A$. Then A is skew-symmetric.

Note that $A^t \neq A$. Then A is not symmetric.

(e) Let
$$B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$
.

Note that
$$B^t = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
.

We have $B^t \neq B$. Then *B* is not symmetric.

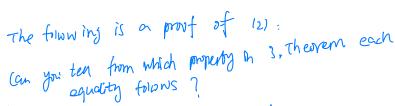
We have $B^t \neq -B$. Then *B* is not skew-symmetric.

(f) Let
$$B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
.

Note that
$$B^t = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
.

We have $B^t \neq B$. Then *B* is not symmetric.

We have $B^t \neq -B$. Then B is not skew-symmetric.



can you find a motion of the symmetric?

6. Lemma (2).

Suppose A is a square matrix. Then:—

- (1) $A + A^t$ is symmetric.
- (2) $A A^t$ is skew-symmetric.

Proof of Lemma (2).

Suppose A is a square matrix.

$$(A - A^{t})^{t} = |A + (-A^{t})|^{t}$$

$$= A^{t} + |A^{t}|^{t}$$

$$= A^{t} - (A^{t})^{t} = A^{t} - A$$

(1) We have $(A + A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$.

Then, by definition of symmetric matrix, $A + A^t$ is symmetric.

(2) We have $(A - A^t)^t = [A + (-A^t)]^t = A^t + (-A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t)$. Then, by definition of skew-symmetric matrix, $A - A^t$ is skew-symmetric.

7. Theorem (3).

Suppose A is a square matrix. Then there are some unique square matrices B, C such that B is symmetric, C is A guide way to prove the theorem;

if A = B + C With B symmetric C stew-symmetric skew-symmetric, and A = B + C.

Proof of Theorem (3).

Suppose A is a square matrix.

- We have two tasks, which are (α) , (β) below:—

 then

 At $= \beta t + C^{\dagger} = \beta C$ (Wy?)

 (α) Conceive some appropriate symmetric matrix, and some appropriate skew-symmetric matrix, respectively [We have two tasks, which are (α) , (β) below:-
- labelled B, C in the subsequent consideration, which we hope will satisfy A = B + C.
- (β) Then we verify for such a pair of matrices B,C two things:— then solving the equations;
 - (1) The equality A = B + C holds indeed.
 - (2) If some symmetric matrix P and some skew-symmetric matrix Q also satisfy A = P + Q, then P = B and Q = C.

We proceed with (α) , and follow up with (β) .

But how to proceed with (α) ?

[Roughwork.

 $\begin{cases}
Az B+C & y B = \frac{A+A^{t}}{2} \\
At = B-C.
\end{cases}$

According to Lemma (2), we have a pair of symmetric matrix and skew-symmetric matrix determined by A alone:—

- $A + A^t$ is a symmetric matrix.
- $A A^t$ is a skew-symmetric matrix.

However, because $(A + A^t) + (A - A^t) = 2A$, they are not the respective B, C that we hope for. But we are getting close.1

3

Define
$$B = \frac{1}{2}(A + A^t)$$
, and $C = \frac{1}{2}(A - A^t)$.

Note that *B* is symmetric, and *C* is skew-symmetric. (Why? Apply Lemma (2) and Theorem (1).)

- We have $B + C = \frac{1}{2}(A + A^t) + \frac{1}{2}(A A^t) = A$.
- Suppose *P* is a symmetric matrix, *Q* is a skew-symmetric matrix, and A = P + Q.

By assumption, $P^t = P$ and $Q^t = -Q$. Then $A^t = (P + Q)^t = P^t + Q^t = P - Q$.

Now we have $2P = (P + Q) + (P - Q) = A + A^t$. Then $P = \frac{1}{2}(A + A^t) = B$.

We also have $2Q = (P + Q) - (P - Q) = A - A^t$. Then $Q = \frac{1}{2}(A - A^t) = C$.