0.1 Transpose, symmetry and skew-symmetry.

0. *Assumed background.*
   - 1.1 *Matrices, matrix addition, and scalar multiplication for matrices.*
   - 1.2 *Matrix multiplication.*

   Abstract. We introduce:—
   - the notion of transpose,
   - the notions of symmetry and skew-symmetry.

   In the *appendix*, we digress onto the notion of definition, theorem, proof, and the format which dictates how they are to be read.

1. **Definition. (Transpose of a matrix.)**

   Let A be an $(m \times n)$-matrix, whose (i, j)-th entry is denoted by a_{ij}.

   The **transpose of** *A is the $(n \times m)$-matrix whose (k, ℓ)-th entry is given by $a_{\ell k}$.*

   It is denoted by A^t.

   We conclude that $\Leftarrow$ (i,j)-th entry of A = (j,i)-th entry of A^t

   Remark. In symbolic terms, what this definition says is:—

   $$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \text{ then } A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

   First thoughts on the equality $(AC)^t = C^tA^t$:

   A: $(m \times n)$

   C: $(n \times p)$

   AC: $(m \times p)$-matrix

   $(AC)^t$: $(p \times m)$-matrix

2. **Example (1). (Transpose of a matrix.)**

   Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}$.

   Then $A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B^t = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$ and $C^t = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

   relation of transpose with addition & multiplication

   (a) Note that $A + B = \begin{bmatrix} 2 & 5 & 3 \\ 2 & 2 & 3 \end{bmatrix}$. Then $(A + B)^t = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$.

   We have $A^t + B^t = \cdots = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$. So $(A + B)^t = A^t + B^t$ (in this example).

   (b) Note that $AC = \cdots = \begin{bmatrix} 4 & 13 \\ 2 & 7 \end{bmatrix}$. Then $(AC)^t = \begin{bmatrix} 4 & 2 \\ 13 & 7 \end{bmatrix}$

   We have $C^tA^t = \cdots = \begin{bmatrix} 4 & 2 \\ 13 & 7 \end{bmatrix}$. So $(AC)^t = C^tA^t$ (in this example).

   A^t $n \times m$

   C^t : $p \times n$

   C^tA^t : $(p \times m)$-matrix

3. **Theorem (1). (Basic properties of transpose.)**

   Suppose A, B are $(m \times n)$-matrices, C is an $(n \times p)$-matrix, and λ is a number. Then:—

   (Not a proof!!)

   (1) *Suppose A is an $(m \times n)$-matrix. Then $(A^t)^t = A$.*

   (2) *Suppose A, B are $(m \times n)$-matrices. Then $(A + B)^t = A^t + B^t$.*

   (3) *Suppose A is an $(m \times n)$-matrix, and λ is a number. Then $(\lambda A)^t = \lambda A^t$.*

   (4) *Suppose A is an $(m \times n)$-matrices, and C is an $(n \times p)$-matrix. Then $(AC)^t = C^tA^t$.*

   Proof of Statement (4) of Theorem (1).

   Suppose A is a $(m \times n)$-matrix, and C is an $(n \times p)$-matrix. (So AC is an $(m \times p)$-matrix, and $(AC)^t$ is a $(p \times m)$-matrix.)

   (By definition, A^t is an $(n \times m)$-matrix, and C^t is a $(p \times n)$-matrix. So C^tA^t is well-defined as a $(p \times m)$-matrix.)

   Denote the (i, j)-th entry of A by a_{ij}. Denote the (k, ℓ)-th entry of C by $c_{k\ell}$.

   Fix any $\ell = 1, 2, \cdots, p$ and $i = 1, 2, \cdots, m$.

- By the definition of matrix multiplication, the (i, ℓ)-th entry of AC is given by $\sum_{j=1}^{n} a_{ij}c_{j\ell}$.

  Then, by the definition of transpose, the (ℓ, i)-th entry of (AC) is given by $\sum_{j=1}^{n} a_{ij}c_{j\ell}$.

- By the definition of transpose, for each $j = 1, 2, \cdots, n$, the (ℓ, j)-th entry of C^t is $c_{j\ell}$, and the (j, i)-th entry of A^t is a_{ij}.

  Then, by the definition of matrix multiplication, the (ℓ, i)-th entry of C^tA^t is given by $\sum_{j=1}^{n} a_{ij}c_{j\ell}$.

Hence $(AC)^t = C^tA^t$.

Proof of Statements (1), (2), (3) of Theorem (1). Exercise. (Imitate what is done above.)

4. **Definition. (Symmetric matrix and skew-symmetric matrix.)**

   Suppose A is an $(n \times n)$-square matrix. Then:—

   (1) *A is said to be* **symmetric** *if and only if $A^t = A$.*

   (2) *A is said to be* **skew-symmetric** *if and only if $A^t = -A$.*

5. **Example (2). (Examples and non-examples on symmetric matrices and skew-symmetric matrices.)**

   (a) The $(n \times n)$-zero matrix is a symmetric matrix. It is also a skew-symmetric matrix.

   (b) The identity matrix is a symmetric matrix. It is not skew-symmetric.

   (c) Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 6 \end{bmatrix}$.

   Note that $A^t = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 6 \end{bmatrix} = A$. Then A is symmetric.

   Note that $A^t \neq -A$. Then A is not skew-symmetric.

   Can you find a matrix A which is both symmetric and skew-symmetric?

   i.e. $\begin{cases} A^t = A \\ A^t = -A \end{cases}$

   (d) Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$.

   Note that $A^t = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -A$. Then A is skew-symmetric.

   Note that $A^t \neq A$. Then A is not symmetric.

   (e) Let $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

   Note that $B^t = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

   We have $B^t \neq B$. Then B is not symmetric.

   We have $B^t \neq -B$. Then B is not skew-symmetric.

   (f) Let $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   Note that $B^t = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   We have $B^t \neq B$. Then B is not symmetric.

   We have $B^t \neq -B$. Then B is not skew-symmetric.

6. **Lemma (2).**

   Suppose A is a square matrix. Then:—

   (1) *$A + A^t$ is symmetric.*

   (2) *$A - A^t$ is skew-symmetric.*

   Proof of Lemma (2).

   Suppose A is a square matrix.

The following is a proof of (2):

Can you tell from which property in 3, Theorem each equality follows?

$$(A - A^t)^t = (A + (-A^t))^t$$
$$= A^t + (-A^t)^t$$
$$= A^t \quad (-(A^t)^t)$$
$$= A^t - (A^t)^t = A^t - A$$

(1) We have $(A+A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$.
Then, by definition of symmetric matrix, $A + A^t$ is symmetric.

(2) We have $(A-A^t)^t = [A + (-A^t)]^t = A^t + (-A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t)$.
Then, by definition of skew-symmetric matrix, $A - A^t$ is skew-symmetric.

7. **Theorem (3).**
Suppose A is a square matrix. Then there are some unique square matrices B, C such that B is symmetric, C is skew-symmetric, and $A = B + C$.

A quick way to prove the theorem:
if $A = B + C$ with B symmetric, C skew-symmetric
then. $A^t = B^t + C^t = B - C$ (why?)

Proof of Theorem (3).
Suppose A is a square matrix.
[We have two tasks, which are (α), (β) below:—

(α) Conceive some appropriate symmetric matrix, and some appropriate skew-symmetric matrix, respectively labelled B, C in the subsequent consideration, which we hope will satisfy $A = B + C$.

(β) Then we verify for such a pair of matrices B, C two things:— then solving the equations:
  (1) The equality '$A = B + C$' holds indeed.
  (2) If some symmetric matrix P and some skew-symmetric matrix Q also satisfy $A = P + Q$, then $P = B$ and $Q = C$.

$$\begin{cases} A = B + C \\ A^t = B - C. \end{cases} \Rightarrow B = \frac{A+A^t}{2}, \quad C = \frac{A-A^t}{2}$$

We proceed with (α), and follow up with (β).
But how to proceed with (α)?]
[*Roughwork.*
According to Lemma (2), we have a pair of symmetric matrix and skew-symmetric matrix determined by A alone:—

- $A + A^t$ is a symmetric matrix.
- $A - A^t$ is a skew-symmetric matrix.

However, because $(A + A^t) + (A - A^t) = 2A$, they are not the respective B, C that we hope for. But we are getting close.]

Define $B = \frac{1}{2}(A + A^t)$, and $C = \frac{1}{2}(A - A^t)$.
Note that B is symmetric, and C is skew-symmetric. (Why? Apply Lemma (2) and Theorem (1).)

- We have $B + C = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) = A$.
- Suppose P is a symmetric matrix, Q is a skew-symmetric matrix, and $A = P + Q$.
  By assumption, $P^t = P$ and $Q^t = -Q$. Then $A^t = (P + Q)^t = P^t + Q^t = P - Q$.
  Now we have $2P = (P + Q) + (P - Q) = A + A^t$. Then $P = \frac{1}{2}(A + A^t) = B$.
  We also have $2Q = (P + Q) - (P - Q) = A - A^t$. Then $Q = \frac{1}{2}(A - A^t) = C$.