0.1 Matrices, matrix addition, and scalar multiplication for matrices.

0. *Abstract.* We introduce:—
   - the notion of matrices, and the notion of equality for matrices,
   - matrix addition, and its properties,
   - the notions of zero matrix and additive inverse,
   - scalar multiplication for matrices, and its properties,
   - presentation of matrices in terms of blocks, and presentation of matrix addition and scalar multiplication in terms of blocks.

1. **Definition. (Matrices.)**
   An $(m \times n)$**-matrix** *(or matrix of size* m *by* n*) with real/complex entries, with* m *rows and* n *columns, is an* $(m \times n)$*-rectangular array*

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{bmatrix}$$

for example ; X_{23} : 2-nd row and 3-rd column

in which the mn *entries* x_{ij}*'s are respectively real/complex numbers.*
Denote such a matrix by X*. Fix any* $k = 1, 2, \cdots, m$*, and any* $\ell = 1, 2, \cdots, n$*.*

   (1) *The* k**-th row** *of the matrix* X *is the 'horizontal' array*

$$\begin{bmatrix} x_{k1} & x_{k2} & x_{k3} & \cdots & x_{kn} \end{bmatrix}.$$

   (2) *The* ℓ**-th column** *of* X *is the 'vertical' array*

$$\begin{bmatrix} x_{1\ell} \\ x_{2\ell} \\ x_{3\ell} \\ \vdots \\ x_{m\ell} \end{bmatrix}.$$

   (3) *The* (k, ℓ)**-th entry** *(or the* (k, ℓ)**-th element***) of* X *is number* $x_{k\ell}$*.*
   (It is where the k*-th row and the* ℓ*-th column of* X *meet.)*

   Further terminologies.
   A $(1 \times n)$*-matrix (with just one row) is also called a* **row vector** *with size* n*.*
   An $(m \times 1)$*-matrix (with just one column) is also called a* **column vector** *with size* m*.*

2. **Example (1).**

   (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a (3×2)-matrix.
   Its first, second and third rows are

$$\begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 5 & 6 \end{bmatrix}$$

   respectively.
   Its first and second columns are

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

   respectively.

   (b) $X = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ is a (4×4)-matrix.

   Another way to present the matrix : $X_{ij} = i + j - 1$ (check it)

   Its first, second, third and fourth rows are

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 & 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 4 & 5 & 6 & 7 \end{bmatrix}$$

   respectively.

Its first, second, third and fourth columns are

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$

respectively.

3. **Example (2).**

   (a) Let $X = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{bmatrix}$.

   The (i, j)-th entry of X is i^j.

   (b) Let a be a real number, and $X = \begin{bmatrix} a & a & a \\ 0 & a & a \\ 0 & 0 & a \end{bmatrix}$. $\Leftarrow$ upper triangular matrix

   Denote the (i, j)-th entry of X by x_{ij}.

   Then $x_{ij} = \begin{cases} a & \text{if} \quad i \leq j \\ 0 & \text{if} \quad i > j \end{cases}$.

   (c) Let $X = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ e & 1 & 0 & \cdots & 0 \\ e^2 & e & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^n & e^{n-1} & e^{n-2} & \cdots & 1 \end{bmatrix}$. $\Leftarrow$ lower triangular matrix

   Denote the (i, j)-th entry of X by x_{ij}.

   Then $x_{ij} = \begin{cases} e^{i-j} & \text{if} \quad i \geq j \\ 0 & \text{if} \quad i < j \end{cases}$.

4. **Definition. (Equality for matrices.)**

   Let A be an $(m \times n)$-matrix, with its (i, j)-th entry being a_{ij} for each i, j.

   Let B be a $(p \times q)$-matrix, with its (k, ℓ)-th entry being b_{ij} for each k, ℓ.

   We say that A, B are **equal (as matrices)** *if and only if:*

   (1) *$m = p$ and $n = q$, and moreover,*

   (2) *$a_{ij} = b_{ij}$ for all i, j.*

   for example, the following 2 matrices are not equal:

   $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

5. **Definition. (Addition for matrices.)**

   Let A, B be $(m \times n)$-matrices with the (i, j)-th entries respectively given by a_{ij}, b_{ij} for each i, j.

   We define the **sum of the matrices** *A, B to be the $(m \times n)$-matrix whose (i, j)-th entry is $a_{ij} + b_{ij}$ for each i, j.*

   It is denoted by $A + B$.

   *(We also read $A + B$ as the '***resultant of** *B* **added to** *A'.)*

   Remark. In symbols, this definition says:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

6. **Example (3). (Addition for matrices.)**

   (a) $\begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} (-1) + 1 & 0 + 2 \\ 2 + (-1) & (-2) + 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$.

   (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 5 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 + 7 & 2 + 5 & 3 + 3 \\ 4 + 5 & 5 + 3 & 6 + 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 6 \\ 9 & 8 & 7 \end{bmatrix}$.

7. **Theorem (1). (Commutativity and associativity of matrix addition.)**

   (1) *Suppose A, B are $(m \times n)$-matrices. Then $A + B = B + A$.*

   (2) *Suppose A, B, C are $(m \times n)$-matrices. Then $A + (B + C) = (A + B) + C$.*

Remark. By virtue of (2), we agree to write '$A + B + C$' for either '$A + (B + C)$' or '$(A + B) + C$'.

8. **Proof of Statement (1) of Theorem (1).** Suppose A, B are $(m \times n)$-matrices. Denote the respective (i, j)-th entries of A, B by a_{ij}, b_{ij} for each i, j.

   Fix any i, j.

   By the definition of matrix addition, the (i, j)-th entry of $A + B$ is $a_{ij} + b_{ij}$.

   Similarly, The (i, j)-th entry of $B + A$ is $b_{ij} + a_{ij}$.

   By the commutativity of addition for real/complex numbers, we have $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.

   Then by the definition of matrix equality, $A + B = B + A$.

   Proof of Statement (2) of Theorem (1). This is left as an exercise.
   (Imitate what is done above, using associativity of addition for real/complex numbers instead.)

9. **Theorem (2). ('Existence and uniqueness' of 'additive identity' for matrices.)**

   There is a unique $(m \times n)$-matrix Z such that for any $(m \times n)$-matrix A, the equality $A + Z = A$ holds.

   Proof of Theorem (2). Let Z be the $(m \times n)$-matrix whose entries are all 0.

   [We intend to verify two things:

   (1) *The equality $A + Z = A$ holds for any $(m \times n)$-matrix A.*

   (2) *If some $(m \times n)$-matrix Y possesses the property '$A + Y = A$ for any $(m \times n)$-matrix A', then $Y = Z$.*

   We proceed with (1), (2) separately.]

   Let A be an $(m \times n)$-matrix with the (i, j)-th entry given by a_{ij} for each i, j.

   (1) For each i, j, the (i, j)-th entry of $A + Z$ is given by $a_{ij} + 0 = a_{ij}$.
   Then by the definition of matrix addition, $A + Z = A$.

   (2) Let Y be an $(m \times n)$-matrix with the (i, j)-th entry given by y_{ij} for each i, j. Suppose $A + Y = A$.
   By the definition of matrix addition, for each i, j, we have $a_{ij} + y_{ij} = a_{ij}$. Then $y_{ij} = 0$.
   Therefore, by the definition of matrix equality, $Y = Z$.

   A more tricky manner to prove (2): If there exists another Y s.t for any A, we have $A + Y = Y + A = A$, then we have $Z + Y = Y + Z = Z$ (replace A by Z above) $(*)$ But we know that for any A, $A + Z = Z + A = A$, Hence replace A by Y, we get $Y + Z = Z + Y = Y$ $(**)$ Then by $(*)$, $(**)$, $Y = Z$.

10. **Definition. (Zero matrix.)**

    *The $(m \times n)$-matrix whose entries are all 0 is called the $(m \times n)$-**zero matrix**.*

    It is denoted by $O_{m \times n}$, (or simply O when no confusion arises).

11. **Theorem (3). ('Existence and uniqueness' of 'additive inverse' for a matrix)**

    Suppose A is an $(m \times n)$-matrix. Then there is a unique $(m \times n)$-matrix C such that $A + C = O_{m \times n}$.

    Proof of Theorem (3). Exercise, imitating the proof of Theorem (2). [We provide the beginning steps below:

    Suppose A is an $(m \times n)$-matrix, with its (i, j)-th entry given by a_{ij} for each i, j.
    Let P be the $(m \times n)$-matrix, with its (i, j)-th entry given by $-a_{ij}$ for each i, j.

    Now imitate the argument for Theorem (2) to verify the statements below:—

    (1) $A + P = O_{m,n}$.

    (2) *If Q is an $(m \times n)$-matrix satisfying $A + Q = O_{m,n}$ then $Q = P$ as matrices.*

    Fill in the detail as an exercise.]

12. **Definition. (Additive inverse, 'matrix subtraction'.)**

    Let A, B be $(m \times n)$-matrices with the (i, j)-th entries respectively given by a_{ij}, b_{ij} for each i, j.

    (a) *The **additive inverse of** A is the $(m \times n)$-matrix whose (i, j)-th entry is given by $-a_{ij}$ for each i, j.*
    It is denoted by $-A$.
    *(We also read $-A$ as '**minus** A'.)*

    (b) *The **difference of** B **from** A is the $(m \times n)$-matrix given by the sum $B + (-A)$.*
    (For each i, j, its (i, j)-th entry is given by $b_{ij} - a_{ij}$.)
    We may write $B + (-A)$ as $B - A$.
    *(We also read $B - A$ as the '**resultant of subtracting** A **from** B'.)*

    $\Leftarrow$ Analogous as the real numbers: $2 - 1 = 2 + (-1)$

13. **Definition. (Scalar multiplication for matrices.)**

Let A be an $(m \times n)$-matrix with real/complex entries, with its (i,j)-th entry given by a_{ij} for each i, j. Let λ be a real/complex number.

The **product of the matrix A by the scalar λ** *is defined to be the $(m \times n)$-matrix whose (i,j)-th entry is λa_{ij} for each i, j. It is denoted by λA.*

(We also read λA as 'the **scalar multiple of A by λ***', or the '***resultant of multiplying the matrix A by the scalar λ***'.)*

Remark. In symbols, this definition says:

$$\lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

14. **Example (4). (Scalar multiplication for matrices.)**

(a) $3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 & 3\cdot 2 & 3\cdot 3 \\ 3\cdot 4 & 3\cdot 5 & 3\cdot 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}.$

(b) $(5\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 4 \end{bmatrix}) + (2\begin{bmatrix} 0 & 8 \\ 7 & 0 \\ 6 & 5 \end{bmatrix}) = \begin{bmatrix} 5 & 0 \\ 0 & 10 \\ 15 & 20 \end{bmatrix} + \begin{bmatrix} 0 & 16 \\ 14 & 0 \\ 12 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 16 \\ 14 & 10 \\ 27 & 30 \end{bmatrix}.$

15. **Theorem (4). (Properties of scalar multiplication for matrices.)**

Suppose A, B are $(m \times n)$-matrices, and λ, μ are scalars. Then:—

(1) $\lambda(A + B) = \lambda A + \lambda B$.

(2) $(\lambda + \mu)A = \lambda A + \mu A$.

(3) $\lambda(\mu A) = (\lambda\mu)A$.

(4) $1A = A$.

(5) $(-1)A = -A$.

(6) $0A = O_{m\times n}$.

Proof of Theorem (4). Exercise. (Imitate the arguments for Theorem (1).)

16. **Presentation of matrices in blocks, introduced through examples.**

Very often, for one reason or another, we like to:—

- visualize various 'rectangular blocks of entries' inside a given matrix as matrices on their own, or
- construct a matrix by putting given matrices of 'smaller sizes' alongside each other.

We introduce this idea through concrete examples. same number of rows.

(a) Let $A_1, A_2, \cdots, A_p$ be matrices each with m rows, and with $n_1, n_2, \cdots, n_p$ columns respectively.
The matrix $[\ A_1 \mid A_2 \mid \cdots \mid A_p\ ]$ stands for the $(m \times (n_1 + n_2 + \cdots + n_p))$-matrix whose columns from left to right are that of $A_1, A_2, \cdots, A_p$ in succession, each from left to right.

Illustration.

Let $A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}$, $A_3 = \begin{bmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{35} & a_{36} \\ a_{45} & a_{46} \end{bmatrix}$.

Then $[\ A_1 \mid A_2 \mid A_3\ ] = \left[\begin{array}{ccc|c|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{array}\right]$.

(b) Let $B_1, B_2, \cdots, B_p$ be matrices each with n columns, and with $m_1, m_2, \cdots, m_p$ rows respectively.

same number of columns.

The matrix $\left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_p \end{array}\right]$ stands for the $((m_1 + m_2 + \cdots + m_p) \times n)$-matrix whose rows from top to bottom are that of $B_1, B_2, \cdots, B_p$ in succession, each from top to bottom.

Illustration.

Let $B_1 = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$, $B_2 = \begin{bmatrix} b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$, $B_3 = \begin{bmatrix} b_{51} & b_{52} & b_{53} & b_{54} \\ b_{61} & b_{62} & b_{63} & b_{64} \end{bmatrix}$.

Then $\left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline B_3 \end{array}\right] = \left[\begin{array}{cccc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ \hline b_{41} & b_{42} & b_{43} & b_{44} \\ \hline b_{51} & b_{52} & b_{53} & b_{54} \\ b_{61} & b_{62} & b_{63} & b_{64} \end{array}\right]$.

(c) The same idea can be extended to the construction of matrices with rows and columns of blocks.

Illustration.

Let $C_{11} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix}$, $C_{12} = \begin{bmatrix} c_{14} \\ c_{24} \\ c_{34} \\ c_{44} \end{bmatrix}$, $C_{13} = \begin{bmatrix} c_{15} & c_{16} \\ c_{25} & c_{26} \\ c_{35} & c_{36} \\ c_{45} & c_{46} \end{bmatrix}$, $C_{21} = \begin{bmatrix} c_{51} & c_{52} & c_{53} \\ c_{61} & c_{62} & c_{63} \\ c_{71} & c_{72} & c_{73} \end{bmatrix}$, $C_{22} = \begin{bmatrix} c_{54} \\ c_{64} \\ c_{74} \end{bmatrix}$, $C_{23} = \begin{bmatrix} c_{55} & c_{56} \\ c_{65} & c_{66} \\ c_{75} & c_{76} \end{bmatrix}$,

Then $\left[\begin{array}{c|c|c} C_{11} & C_{12} & C_{13} \\ \hline C_{21} & C_{22} & C_{32} \end{array}\right] = \left[\begin{array}{ccc|c|cc} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ \hline c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \\ c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} \end{array}\right]$.

17. **Theorem (5).**

Let A_{ij}, B_{ij} be matrices for each $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$.

Suppose that for each $i = 1, 2, \cdots, p$, the matrices $A_{i1}, A_{i2}, \cdots, A_{iq}, B_{i1}, B_{i2}, \cdots, B_{iq}$ have the same number of rows.

Suppose that for each $j = 1, 2, \cdots, q$, the matrices $A_{1j}, A_{2j}, \cdots, A_{pj}, B_{1j}, B_{2j}, \cdots, B_{pj}$ have the same number of columns.

Define $A = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1q} \\ \hline A_{21} & A_{22} & \cdots & A_{2q} \\ \hline \vdots & \vdots & & \vdots \\ \hline A_{p1} & A_{p2} & \cdots & A_{pq} \end{array}\right]$, *and* $B = \left[\begin{array}{c|c|c|c} B_{11} & B_{12} & \cdots & B_{1q} \\ \hline B_{21} & B_{22} & \cdots & B_{2q} \\ \hline \vdots & \vdots & & \vdots \\ \hline B_{p1} & B_{p2} & \cdots & B_{pq} \end{array}\right]$.

Then $A + B = \left[\begin{array}{c|c|c|c} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1q}+B_{1q} \\ \hline A_{21}+B_{21} & A_{22}+B_{22} & \cdots & A_{2q}+B_{2q} \\ \hline \vdots & \vdots & & \vdots \\ \hline A_{p1}+B_{p1} & A_{p2}+B_{p2} & \cdots & A_{pq}+B_{pq} \end{array}\right]$.

Moreover, $\lambda A = \left[\begin{array}{c|c|c|c} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1q} \\ \hline \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2q} \\ \hline \vdots & \vdots & & \vdots \\ \hline \lambda A_{p1} & \lambda A_{p2} & \cdots & \lambda A_{pq} \end{array}\right]$

for each number λ.

Proof of Theorem (5). Omitted. (This is omitted not because it is difficult, but because it is a tedious and straightforward exercise in book-keeping.)

In the course, I didn't talk about all the examples here. It's useful to check them carefully.

18. **Illustrations of the content of Theorem (5).**

(a) Let $A = \left[\begin{array}{ccc|c|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{array}\right]$, $B = \left[\begin{array}{ccc|c|cc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \end{array}\right]$.

Let $A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}$, $A_3 = \begin{bmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{35} & a_{36} \\ a_{45} & a_{46} \end{bmatrix}$.

Let $B_1 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$, $B_2 = \begin{bmatrix} b_{14} \\ b_{24} \\ b_{34} \\ b_{44} \end{bmatrix}$, $B_3 = \begin{bmatrix} b_{15} & b_{16} \\ b_{25} & b_{26} \\ b_{35} & b_{36} \\ b_{45} & b_{46} \end{bmatrix}$.

Then we have $A = \left[\begin{array}{c|c|c} A_1 & A_2 & A_3 \end{array}\right]$, $B = \left[\begin{array}{c|c|c} B_1 & B_2 & B_3 \end{array}\right]$, and

$A_1 + B_1 = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \\ a_{41}+b_{41} & a_{42}+b_{42} & a_{43}+b_{43} \end{bmatrix}$, $A_2 + B_2 = \begin{bmatrix} a_{14}+b_{14} \\ a_{24}+b_{24} \\ a_{34}+b_{34} \\ a_{44}+b_{44} \end{bmatrix}$, $A_3 + B_3 = \begin{bmatrix} a_{15}+b_{15} & a_{16}+b_{16} \\ a_{25}+b_{25} & a_{26}+b_{26} \\ a_{35}+b_{35} & a_{36}+b_{36} \\ a_{45}+b_{45} & a_{46}+b_{46} \end{bmatrix}$.

So $A + B = \left[\begin{array}{c|c|c} A_1+B_1 & A_2+B_2 & A_3+B_3 \end{array}\right]$ indeed.

(b) Let $C = \left[\begin{array}{cccc} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ \hline c_{41} & c_{42} & c_{43} & c_{44} \\ \hline c_{51} & c_{52} & c_{53} & c_{54} \\ c_{61} & c_{62} & c_{63} & c_{64} \end{array}\right]$.

Let $C_1 = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$, $C_2 = \begin{bmatrix} c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$, $C_3 = \begin{bmatrix} c_{51} & c_{52} & c_{53} & c_{54} \\ c_{61} & c_{62} & c_{63} & c_{64} \end{bmatrix}$.

Then $C = \left[\begin{array}{c} C_1 \\ \hline C_2 \\ \hline C_3 \end{array}\right]$.

For each number λ, we have $\lambda C_1 = \begin{bmatrix} \lambda c_{11} & \lambda c_{12} & \lambda c_{13} & \lambda c_{14} \\ \lambda c_{21} & \lambda c_{22} & \lambda c_{23} & \lambda c_{24} \\ \lambda c_{31} & \lambda c_{32} & \lambda c_{33} & \lambda c_{34} \end{bmatrix}$, $\lambda C_2 = \begin{bmatrix} \lambda c_{41} & \lambda c_{42} & \lambda c_{43} & \lambda c_{44} \end{bmatrix}$, $\lambda C_3 = \begin{bmatrix} \lambda c_{51} & \lambda c_{52} & \lambda c_{53} & \lambda c_{54} \\ \lambda c_{61} & \lambda c_{62} & \lambda c_{63} & \lambda c_{64} \end{bmatrix}$.

So $\lambda C = \left[\begin{array}{c} \lambda C_1 \\ \hline \lambda C_2 \\ \hline \lambda C_3 \end{array}\right]$

(c) Let $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, B_{21}, B_{22}$ be matrices.
Suppose that:—

- the number of rows of $A_{11}, A_{12}, B_{11}, B_{12}$ are the same,
- the number of rows of $A_{21}, A_{22}, B_{21}, B_{22}$ are the same,
- the number of columns of $A_{11}, A_{21}, B_{11}, B_{21}$ are the same, and
- the number of column of $A_{12}, A_{22}, B_{12}, B_{22}$ are the same.

Define $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$.

Then $A + B = \left[\begin{array}{c|c} A_{11} + B_{11} & A_{12} + B_{12} \\ \hline A_{21} + B_{21} & A_{22} + B_{22} \end{array}\right]$.

Moreover, for each $\alpha \in \mathbb{R}$, $\alpha A = \left[\begin{array}{c|c} \alpha A_{11} & \alpha A_{12} \\ \hline \alpha A_{21} & \alpha A_{22} \end{array}\right]$.