MATH1010G University Mathematics
Week 2: Limits of Sequences

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Review (key concepts)

- set / set operations: union, intersection, relative complement
- radian, trigonometric functions
- trigonometric identities for sin / cos
- **function / piecewise defined function (domain / range)**
- injective / surjective / bijective / inverse
- composition
- odd, even, periodic
- principle of mathematical induction
- **sequence**

Partial fractions

How shall we find ”parts” that make the single fractions ?

Factorize $\frac{5x-4}{x^2-x-2}$.

The key is to properly factorize the bottom

- linear, irreducible quadratic factors

$$\frac{Bx + C}{x^2 + 1}$$

- factors with exponent: partial fraction for each exponent from 1 up

$$\frac{1}{(x+1)^3}: \quad \frac{A_1}{x+1}, \frac{A_2}{(x+1)^2}, \frac{A_3}{(x+1)^3}$$

Factorize $\frac{x^2+15}{(x+1)^2(x^2+3)}$

sequence of real numbers

Definition

A sequence of real numbers $\{a_n\}$ is a function $f : \mathbb{Z}^+ \to \mathbb{R}$: $a_n = f(n)$

Example

sequences with patterns

- $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, ... in general $a_n = 2^{n-1}$, $n \in \mathbb{Z}^+$
- $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, ... in general $a_n = \frac{1}{n}$, $n \in \mathbb{Z}^+$
- $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, ... in general $a_n = (-1)^n$, $n \in \mathbb{Z}^+$

Limits of sequences:

Definition (informal)

Let $\{a_n\}$ be a sequence of real numbers. If n is getting larger and larger, a_n is getting closer and closer to $L \in \mathbb{R}$. Then L is said to be the limit of the sequence $\{a_n\}$, denoted by

$$\lim a_n = L,$$

and the sequence $\{a_n\}$ is said to converge to L.

Example

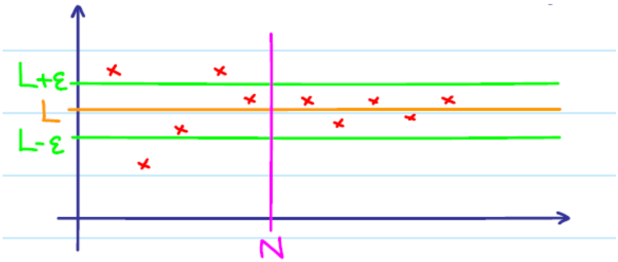
- $\lim \frac{1}{n} = 0$. (However $\frac{1}{n} \neq 0 \quad \forall n \in \mathbb{Z}^+$.)
- $\lim 2^{n-1}$ does not exist. (But some still write $\lim_{n\to\infty} 2^{n-1} = +\infty$ or say 2^{n-1} diverges to $+\infty$.)
- $\lim(-1)^n$ does not exist.

formal definition of limit

Definition ($\epsilon - N$)

Let $\{a_n\}$ be a sequence of real numbers and $L \in \mathbb{R}$. L is said to be the limit of the sequence $\{a_n\}$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}^+$ s.t.

$$|a_n - L| < \epsilon, \quad \forall n \geq N.$$



Meaning: No matter how small ϵ is given, we can find an $N \in \mathbb{Z}^+$ such that the tail (a_n with $n \geq N$) of the sequence lies in the $\epsilon-$tunnel ($\epsilon-$neighbourhood of L)

Theorem

- If $a_n = k$ for all $n \in \mathbb{Z}^+$ (constant seq.), then $\lim_{n\to\infty} a_n = k$.
- If $k > 0$ and $a_n = n^{-k} = \frac{1}{n^k}$ for all $n \in \mathbb{Z}^+$, then $\lim_{n\to\infty} a_n = 0$.
- If $-1 < a < 1$, then $\lim_{n\to\infty} a^n = 0$.

All the above are obvious, but we need to verify the $\epsilon - N$ definition which can be hard.

algebraic properties of limits:

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If

$$\lim a_n = L \quad \text{and} \quad \lim b_n = M,$$

then

(i) $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M.$

(ii) $\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M.$

(iii) $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM.$

(iv) If $M \neq 0$, $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} = \frac{L}{M}$.

(very important assumption for applying the theorem)

Example

Find $\lim \frac{2}{n} + 3$.

Example

Find $\displaystyle\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$.

Exercise

Find $\lim \frac{3n+1}{n^2-2n}$, $\lim \frac{n^3+2n}{2n^2+1}$(if exists).

Conclusion

If $p(x)$ and $q(x)$ are polynomials.

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 \text{ (with } a_m \neq 0, deg(p) = m)$$
$$q(x) = b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0 \text{ (with } b_k \neq 0, deg(q) = k)$$

then

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} \pm\infty & \text{if } m > k, \\ \frac{a_m}{b_m} & \text{if } m = k, \\ 0 & \text{if } m < k. \end{cases}$$

Example

$$\lim_{n\to\infty}\frac{3n-1}{\sqrt{4n^2+2n}}=\lim_{n\to\infty}\frac{3-\frac{1}{n}}{\sqrt{4+\frac{2}{n}}}=\frac{3}{2}.$$

$$\begin{aligned}\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}&=\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\\&=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0.\end{aligned}$$

Example

Find $\lim \frac{2^n}{n}$.

Question: Can we say $\frac{2^n}{n} = \frac{1}{n} \cdot 2^n$ and $\lim \frac{1}{n} = 0$, so

$$\lim \frac{2^n}{n} = 0?$$

Absolutely NOT! Since $\lim\limits_{n\to\infty} 2^n$ does not exist, property (iii) can not be applied!

exercise

Find the limits, if they exist

$$\lim \frac{3n - \sqrt{4n^2 + 1}}{2n + \sqrt{8n^2 + 1}}, \quad \lim n - \sqrt{n^2 - 4n + 2}, \quad \lim \frac{\ln(n^4 + 1)}{\ln(n^5 + 1)}$$

Sandwich theorem / Squeeze theorem

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If

$$a_n \le b_n \le c_n, \quad \forall n \in \mathbb{Z}^+ \quad \text{and} \quad \lim a_n = \lim c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L.$$

Geometrical meaning:

In fact, the result is still true if $a_n \le b_n \le c_n$ for all $n \ge n_0$.

Idea: Estimate a sequence $\{b_n\}$ that we do not understand very well by sequences $\{a_n\}$ and $\{c_n\}$ that we understand well.

Example

Find $\lim \frac{1}{n} \sin n$.

Note: $-\frac{1}{n} \leq \frac{1}{n} \sin n \leq \frac{1}{n}$ for all $n \in \mathbb{Z}^+$ and $\lim -\frac{1}{n} = \lim \frac{1}{n} = 0$.
By the sandwich theorem $\lim \frac{1}{n} \sin n = 0$.

Exercise

Prove that $\lim \frac{(-1)^n}{n} = 0$.

Example

Find $\lim \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

Exercise

If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = -1$, $\lim_{n\to\infty} c_n = 1$, can we conclude

$$-1 \leq \lim_{n\to\infty} b_n \leq 1?$$

Exercise

Prove that $\lim_{n\to\infty} \frac{\sin n + (-1)^n}{n} = 0$.

Hint: $-2 \leq \sin n + (-1)^n \leq 2$.

Theorem

Let $\{a_n\}$ be a sequence of real numbers. Then

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0.$$

a result concerning a product of two sequences:

Theorem

If $\{a_n\}$ *and* $\{b_n\}$ *be sequences of real numbers such that* $\lim a_n = 0$ *and* $\{b_n\}$ *is bounded, then* $\lim a_n b_n = 0$.

- $-|a_n| \leq a_n \leq |a_n| \ \forall n \in \mathbb{Z}^+$.
- $\{b_n\}$ is bounded $\Rightarrow \exists M > 0$ such that $|b_n| \leq M \quad \forall n \in \mathbb{Z}^+$.

Therefore $-M|a_n| \leq a_n b_n \leq M|a_n| \quad \forall n \in \mathbb{Z}^+$.

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0, \text{ so } \lim_{n\to\infty} -M|a_n| = \lim_{n\to\infty} M|a_n| = 0.$$

Hence, by sandwich theorem, $\lim a_n b_n = 0$.

Theorem

Let $\alpha \in \mathbb{R}$*, we have* $\lim \frac{\alpha^n}{n!} = 0$.

- If $|\alpha| \leq 1$ the result is not surprising.
- If $|\alpha| > 1$ α^n grows to ∞ (if $\alpha < -1$ it is oscillating) as n grows to ∞. However, this theorem says that $n!$ grows faster than α^n.

Exercise

Let

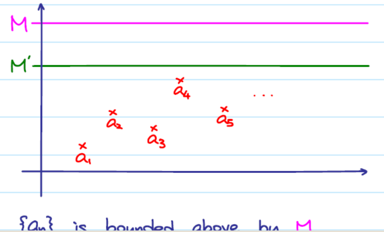
$$a_n = \frac{1}{n^3 + 1^2} + \ldots + \frac{1}{n^3 + n^2}$$

Find $\lim a_n$

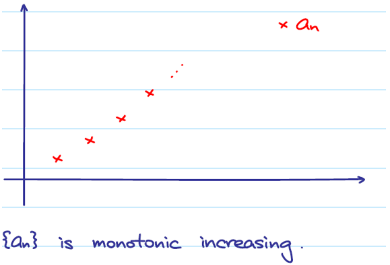
Monotone convergence theorem

Let $\{a_n\}$ be a sequence of real numbers.

- $\{a_n\}$ is bounded above if $\exists M \in \mathbb{R}$ s.t. $a_n \leq M \ \forall n \in \mathbb{Z}^+$.
- $\{a_n\}$ is bounded below if $\exists M \in \mathbb{R}$ s.t. $a_n \geq M \ \forall n \in \mathbb{Z}^+$.
- $\{a_n\}$ is bounded if $\exists M \in \mathbb{R}$ s.t. $|a_n| \leq M \quad \forall n \in \mathbb{Z}^+$.
  bounded = both bounded above and below.



- $\{a_n\}$ is monotonic increasing if $a_{n+1} \geq a_n \ \forall n \in \mathbb{Z}^+$.
- $\{a_n\}$ is monotonic decreasing if $a_{n+1} \leq a_n \ \forall n \in \mathbb{Z}^+$.
- $\{a_n\}$ is monotonic if it is either monotonic increasing or decreasing.



Example

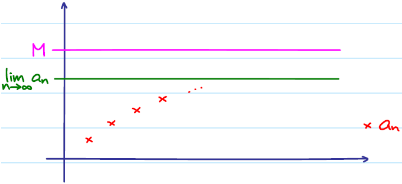
- $\lim_{n\to\infty} 2^{n-1}$ does NOT exist.
  (monotonic but not bounded) monotonic $\not\Rightarrow$ convergent.
- $\lim_{n\to\infty} (-1)^n$ does NOT exist.
  (bounded but not monotonic) bounded $\not\Rightarrow$ convergent.

How about combining them together?

Monotone Convergence Theorem

If $\{a_n\}$ is bounded above (below) and monotonic increasing (decreasing), then $\lim_{n\to\infty} a_n$ exists.

Geometrical meaning :



Caution :

$\{a_n\}$ is bounded above by M .

but $\lim_{n\to\infty} a_n$ is NOT necessary to be M.

Example

Let $\{a_n\}$ be a sequence of positive real numbers defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 + \frac{a_n}{1 + a_n} \quad (n \geq 1).$$

Does $\lim_{n\to\infty} a_n$ exist?

Exercise

Let $\{a_n\}$ be a sequence of positive real numbers defined by

$$a_1 = 2, \quad \text{and} \quad a_{n+1} = 1 + \frac{a_n}{1 + a_n} \quad (n \geq 1).$$

Does $\lim_{n\to\infty} a_n$ exist?

Constant e:

Consider a number $(1+\frac{1}{m})^n$ which depends on both m and n.

fix m, say $m = 100$, n is getting larger and larger.

n	10	100	1000	$n \to \infty$
$(1+\frac{1}{m})^n$	1.01^{10}	1.01^{100}	1.01^{1000}	$(1+\frac{1}{m})^n \to \infty$

fix n, say $n = 100$, m is getting larger and larger.

m	10	100	1000	$n \to \infty$
$(1+\frac{1}{m})^n$	1.1^{100}	1.01^{100}	1.001^{1000}	$(1+\frac{1}{m})^n \to 1$

Constant e:

How about setting $m = n$ and let them larger and larger?

$\lim_{n\to\infty}(1+\frac{1}{n})^n$ exists? Something between 1 and ∞.

n	10	100	1000	$n \to \infty$
$(1+\frac{1}{n})^n$	1.1^{10} ≈ 2.59374	1.01^{100} ≈ 2.70481	1.001^{1000} ≈ 2.71692	2.71828... limit exists: e

Theorem

Let $\{a_n\}$ be a sequence of real numbers defined by

$$a_n = (1 + \tfrac{1}{n})^n.$$

Prove that

- $\{a_n\}$ is monotonic increasing.
- $\{a_n\}$ is bounded above.

Claim: the sequence $\{a_n\}$ is monotonically increasing and bounded above. Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \quad \text{exists.}$$

$\lim \left(1 + \frac{1}{n}\right)^n$ exists and the limit is called e.

summary

- partial fractions
- limit of sequence
- algebraic properties of limits
- sandwich theorem
- monotone convergence theorem