MATH1010G University Mathematics
Week 1: Preliminaries

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Chinese University of Hong Kong

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Outline

topics covered

1. Preliminary (Week 1)
2. Limits and Continuity (Week 2-4)
3. Differentiation (Week 5-6)
4. Applications of Differentiation (Week 7-8)
5. Taylor's Theorem and L'Hôpital's rule (Week 9)
6. Indefinite and Definite Integration (Week 10-13)

Assessment:

- Homework 10 sets (10%) (Total score will be multiplied by 1.25)
- WeBWorK 6 sets (10%)
- Midterm Examination (30%)
- Final Examination (50%)

- HW released on Monday, and submitted later next Thursday. Grading via Gradescope only (don't send them to me or TAs). No makeup for homeworks
- Mid-term exam: March 14 (after reading week)
- Tutorial: Jason Choy
- Tutorial: Thurs. 5:30 - 6:15 pm

Honesty in Academic Work

The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University. Although cases of cheating or plagiarism are rare at the University, everyone should make himself / herself familiar with the content of the following website:

`http://www.cuhk.edu.hk/policy/academichonesty/`

and thereby help avoid any practice that would not be acceptable.

Notation

Set: collection of objects, called elements

- $\subset$: subset
- $\in$: belongs to

Example

$S = \{1, 2, 3\}$, $T = \{1, 2, 3, 4\}$

common notations : $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\emptyset$, $[a, b]$, $[a, b)$, $[a, \infty)$

Example

the set of all positive even numbers

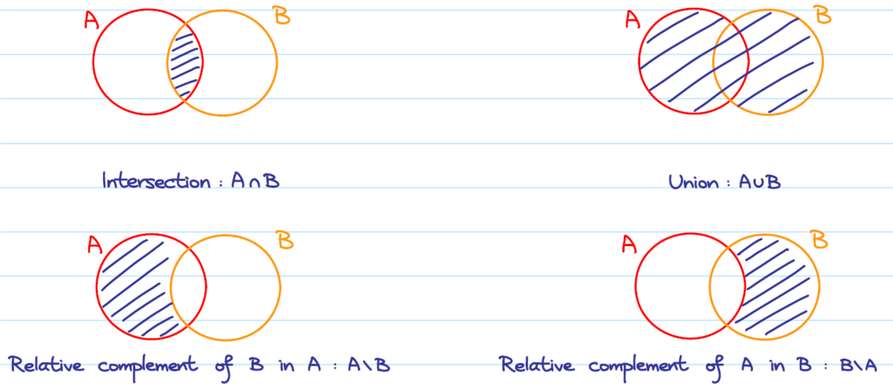
$$\{2, 4, 6, \ldots\} = \{2m : m \in \mathbb{Z}^+\}$$

i.e., this set conissts of elements of the form $2m$ such that $m \in \mathbb{Z}^+$

Exercise

Describe the set all positive odd numbers.

Set operations



Example

Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3\}$. Find $A \cap B$, $A \cup C$, $A \setminus B$, $B \setminus A$

Example

$\mathbb{R} \setminus \{2\}$: the set of all real numbers except 2 (cannot write $\mathbb{R} \setminus 2$!)

Example

Solve $x^2 > 1$

more useful notation:

- $\forall$: for all
- $\exists$: there exists (at least one)
- $\exists!$: there exists a unique
- $\Rightarrow$: implies
- $\Leftrightarrow$: if and only if (iff, equivalent to)
- s.t.: such that

Example

$\forall y \in (0, \infty), \exists x \in \mathbb{R}$ s.t. $x^2 = y$

Example

$\forall y \in (0, \infty), \exists! x \in (0, \infty)$ s.t. $x^2 = y$

Example

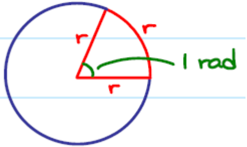
Let $x > 0$.

- $y = \sqrt{x} \Rightarrow y^2 = x$ (true)
- $y^2 = x \Rightarrow y = \sqrt{x}$ (false)

Trigonometry

a second unit of measurement of angle (radian)

When the length of an arc equals to the radius, the angle suspended is defined as 1 radian. In particular, $2\pi\mathrm{rad} = 360^\circ$. From now on, we use only radian.



Exercise

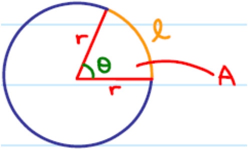
- $\pi\mathrm{rad} =$??
- ?? $= 90^\circ$
- ?? $= 60^\circ$

area of sector $A \propto \theta$:

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \quad \Rightarrow \quad A = \frac{r^2}{2}\theta$$

arclength $\ell \propto \theta$:

$$\frac{\ell}{2\pi r} = \frac{\theta}{2\pi} \quad \Rightarrow \quad \ell = r\theta$$

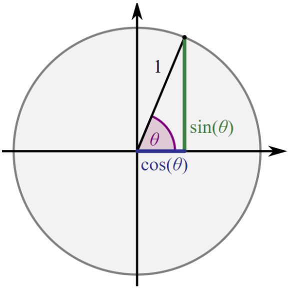


definitions of trigonometric functions

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}$$



basic facts:

$$\sin 0 = 0, \ \sin\frac{\pi}{6} = \frac{1}{2}, \ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \ \sin\frac{\pi}{2} = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

Theorem

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

compound angle formula:

- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

double angle formula

- $\sin(2\alpha) = 2\sin\alpha\cos\alpha$
- $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$
- $\tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha}$

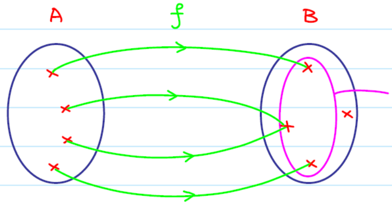
product to sum formula

- $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$
- $-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$
- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

sum to product formula (put $\alpha = \frac{A+B}{2}$, $\beta = \frac{A-B}{2}$)

- $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
- $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
- $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
- $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

A function is a rule that assigns to each element in a set A exactly one element in a set B.



set A : domain (input)

set B : codomain (output)

$\text{range}(f) \subseteq B$: range of f

$\text{range}(f) = f(A) := \{ f(x) \in B : x \in A \}$

($:=$ defined by)

A function f from A to B is denoted by $f : A \to B$

Example

- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, range$(f) = [0, \infty)$
- $f : [-1, 2)$ defined by $f(x) = x^2$, range$(f) = [0, 4)$
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4$, i.e., $y = x^2 + 4$, range: $[4, \infty)$

Example

Find the maximum domain of $f(x) = \frac{2x}{x^2 - 7x}$.

Exercise

If $f(x) = \sqrt{x^2 - 4x + 3}$, find the (maximum) domain of f.
Note that $f(x)$ is a well defined function if $x^2 - 4x + 3 \geq 0$

Exercise

Find the (maximum) domain of $f(x) = \frac{1}{\sqrt{x^2-4x+3}}$.
Note that $f(x)$ is well defined if $x^2 - 4x + 3 > 0$.

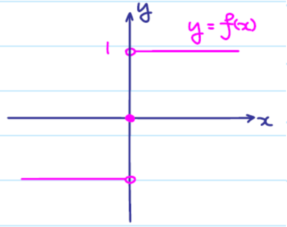
piecewise defined functions

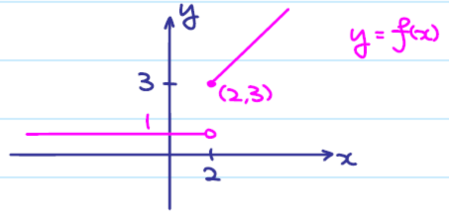
Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \text{ and } f(x) = \begin{cases} x+1, & \text{if } x \geq 2 \\ 1, & \text{if } x < 2 \end{cases}$$

Sketch the functions, and find their ranges?





Exercise

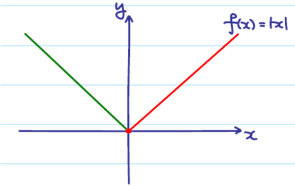
Sketch the graph of $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x + 1, & \text{if } x > 1 \\ 0, & \text{if } 0 \leq x \leq 1 \\ -x^2 & \text{if } x < 0 \end{cases}$$

What is the range of f?

absolute value function $f(x) = |x| = \sqrt{x^2}$
e.g., $|3| = 3$, $|0| = 0$, $|-3| = 3$
$|x|$ as a piecewise defined function

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



Example

What is the graph of $f(x) = 2x + |x - 1|$?

exponential and logarithmic functions

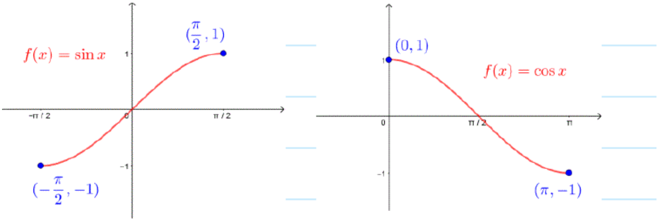
- $y = a^x$, with $a > 0$
  Think: if $a = -1$, when $x = \frac{1}{2}$, $y = a^x = \sqrt{-1}$!
  $a^x > 0$ for any $a > 0$ and $x \in \mathbb{R}$
- $y = \log_a x$ with $a > 1$ or $0 < a < 1$
  Note: $y = \log_a x$ is well defined when $a > 1$ or $0 < a < 1$!
  By definition, if $y = a^x$, then $\log_a y = x$

facts about exponential and logarithmic functions

- $\log_a M + \log_a N = \log_a(MN)$
- $\log_a M - \log_a N = \log_a \frac{M}{N}$
- $\log_a M^n = n \log_a M$
- $\log_a x = \frac{\log_b x}{\log_b a}$ (change of base)
- $e = 2.71828...$ (explain later)
  we write $\log_e x$ as $\ln x$ (natural log function)
- a^x and $\log_a x$ are inverse to each other, i.e., $a^{\log_a x} = x$ and $\log_a a^x = x$

injective and surjective functions

- injective: every $y \in \text{range}(f)$ comes from exactly one $x \in A$
- surjective: every $y \in B$ comes from at least one $x \in A$



Definition

Let $f : A \to B$ be a function.

(i) f is said to be injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
(once the outputs are the same, the inputs must be the same!)

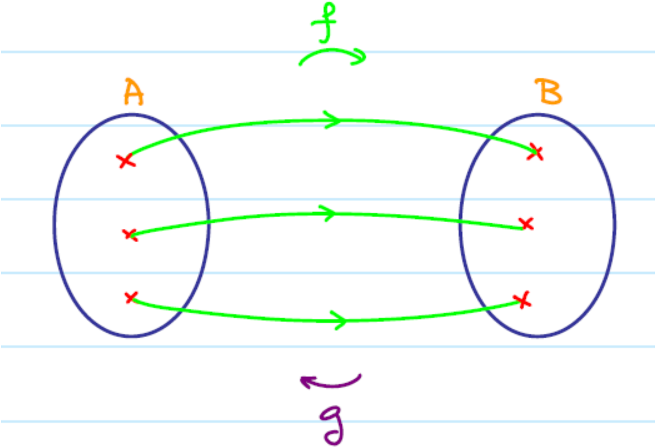
(ii) f is said to be surjective if $\forall y \in B, \exists x \in A$ s.t. $f(x) = y$ i.e. $f(A) = B$

(iii) If f is both injective and surjective, then it is said to be bijective.

Example

Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ is a bijective function.

inverse of a function: intuitive idea



Definition

Let $f : A \to B$ be a function. If $g : B \to A$ is a function such that

$$g(f(x)) = x, \ \ \forall x \in A \text{ and } f(g(y)) = y \ \forall y \in B.$$

Then g is said to be an inverse of f.

- Once an inverse of f exists, it is unique, so we denote it by f^{-1}.
- f has an inverse if and only if f is bijective.

Example

	f	injective	surjective
$f : \mathbb{R} \to \mathbb{R}$	$\sin x$	??	??
$f : \mathbb{R} \to [-1, 1]$	$\sin x$	??	??
$f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$	$\sin x$	??	??

Example

$f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ defined by $f(x) = \sin x$ is bijective, so we can define arcsin $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ s.t.

$$\sin^{-1}(\sin x) = x, \quad \forall x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \quad \text{and} \quad \sin(\sin^{-1} y) = y, \quad \forall y \in [-1, 1]$$

Furthermore,

- $\cos : [0, \pi] \to [-1, 1]$ is bijective, and $\cos^{-1} : [-1, 1] \to [0, \pi]$
- $\tan : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ is bijective, and $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

Example

Let $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$. Note that f is bijective, so $f^{-1} : (0, \infty) \to \mathbb{R}$ can be defined and it is denoted by $f^{-1}(x) = \ln x$.

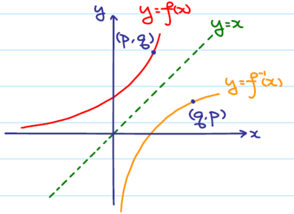
- $f^{-1}(f(x)) = \ln e^x = x, \quad \forall x \in \mathbb{R}$
- $f(f^{-1}(y)) = e^{\ln y} = y, \quad \forall y \in (0, \infty)$

The graph of f^{-1} is the reflection of the graph of f along $y = x$.
Reason: (p, q) lies on the graph of f, i.e.,

$$f(p) = q$$

$$\Leftrightarrow f^{-1}(q) = p$$

i.e. (q, p) lies on the graph of f^{-1}.



Odd and periodic functions

Definition

Let $D \subset \mathbb{R}$.

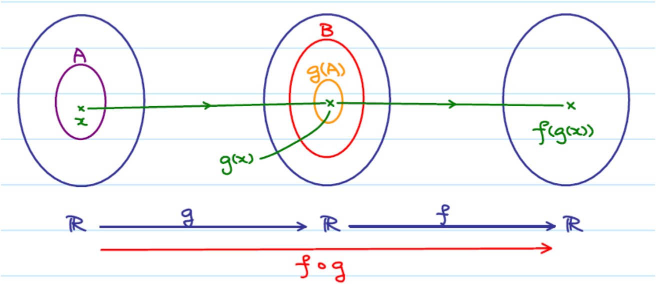
- $f : D \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all $x \in D$.
  property: the graph is symmetric along y-axis
- odd if $f(-x) = -f(x)$ for all $x \in D$
  property: the graph is symmetric about $(0, 0)$

Exercise

If $f : D \to \mathbb{R}$ is an odd function and $0 \in D$. Show $f(0) = 0$

periodic if there exists $T > 0$ such that $f(x) = f(x + T)$ for all $x \in D$
if T is the least positive real number with the above property, T is

composite function: If $f : B \to \mathbb{R}$, and $g : A \to \mathbb{R}$ are functions such that $g(A) \subset B$, we can construct a new function $f \circ g : A \to \mathbb{R}$ which is defined by $f \circ g(x) = f(g(x))$ for all $x \in \mathbb{A}$, which is said to be a composite function "f composite with g"



Example

Let $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$, and let $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 1 + \sin x$. Find $f \circ g$ and $g \circ f$

Example

Let $f(x) = (x^2 + 1)^4$. Then f can be realized as a composite function: $h(x) = x^2 + 1$ and $g(x) = x^4$. Then

$$(g \circ h) = g(x^2 + 1) = (x^2 + 1)^4 = f(x)$$

Mathematical induction

Example

Let $P(n)$ be the statement

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{Z}^+$$

- $P(1)$: $1 = \frac{1(1+1)}{2}$ which is true
- $P(2)$: $1 + 2 = \frac{2(1+2)}{2}$ which is true
- $P(3)$: $1 + 2 + 3 = \frac{3(3+1)}{2}$ which is true

How about $P(4)$, $P(5)$, ... ? We cannot check all one by one!

Theorem (Principle of Mathematical induction)

Let $P(n)$ be a statement where n is a positive integer. Suppose that (i) $P(1)$ is true. (ii) $P(k)$ is true implies that $P(k+1)$ is true. Then $P(n)$ is true for all positive integers n.

idea: The second condition says that if the previous statement is true, then the next one would also be true. From (i), $P(1)$ is true. From (ii), the next statement $P(2)$ is also true. Again, the next statement $P(3)$ is also true, and so on.

Let $P(n)$ be the statement that

$$1 + 3 + \ldots + (2n - 1) = n^2, \quad n \in \mathbb{Z}^+$$

Let $P(n)$ be the statement

$$\sum_{r=1}^{n} \cos r\theta = \frac{\cos \frac{(n+1)\theta}{2} \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}, \quad n \in \mathbb{Z}^{+}$$

sequence of real numbers

Definition

A sequence of real numbers $\{a_n\}$ is a function $f : \mathbb{Z}^+ \to \mathbb{R}$: $a_n = f(n)$

Example

Let $a_1 = 2$, $a_2 = \pi$, $a_3 = \sqrt{3}$, ... or write as $\{2, \pi, \sqrt{3}, ...\}$ (No pattern)

Example

sequences with patterns

- $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, ... in general $a_n = 2^{n-1}$, $n \in \mathbb{Z}^+$
- $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, ... in general $a_n = \frac{1}{n}$, $n \in \mathbb{Z}^+$
- $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, ... in general $a_n = (-1)^n$, $n \in \mathbb{Z}^+$

Example

recursive sequence: let $\{a_n\}$ be a sequence of real numbers, defined by $a_1 = 1$, and $a_{n+1} = a_n^2 + 2$ for $n \geq 1$. Then
$\{a_n\} = \{1, 3, 11, 123, \ldots\}$

Summary (key concepts)

- set / set operations: union, intersection, relative complement
- radian, trigonometric functions
- trigonometric identities for sin / cos
- function / piecewise defined function (domain / range)
- injective / surjective / bijective / inverse
- composition
- odd, even, periodic
- principle of mathematical induction
- sequence