MATH1010G University Mathematics
Week 9: Integration

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Indefinite integration

antiderivatives

A function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$. The process of finding antiderivative is called indefinite integration.

Example

let $f(x) = 2x$ and $F(x) = x^2$

Since $F'(x) = f(x)$, $F(x)$ is an anti derivative of $f(x)$. However, consider $F(x) = x^2 + C$, where C is a constant. We still have $F'(x) = f(x)$. Therefore, antiderivative of $f(x)$ is NOT unique.

Theorem

If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$. Then $F(x)$ and $G(x)$ differ by a constant.

Suppose $F'(x) = G'(x) = f(x)$. Let $H(x) = F(x) - G(x)$. Then $H'(x) = F'(x) - G'(x) = 0$. Hence $H(x)$ is a constant function: $F(x) = G(x) + C$. Therefore, antiderivative of f is unique up to a constant.

Example

$f(x) = 2x$ and $F(x) = x^2$

Then $F'(x) = f(x)$, so $F(x) = x^2$ is an antiderivative of $f(x) = 2x$. and all antiderivative of $f(x)$ must be of the form $x^2 + c$.

If $F(x)$ is an antiderivative of $f(x)$, we write

$$\int f(x)dx = F(x) + c$$

- $\int$ is the integral function.
- $f(x)$ is the integrand.
- dx is the variable of integration.

Example

$\int 2xdx = x^2 + C$

Rules of Indefinite Integration

Theorem

- $\int k dx = kx + c$, for a constant k.
- $\int x^n dx = \frac{1}{n+1} x^{n+1} + c$, for all $n \neq -1$
- $\int \frac{1}{x} dx = \ln |x| + c$
- $\int e^x dx = e^x + c$
- $\int \cos x dx = \sin x + c$
- $\int \sin x dx = -\cos x + c$
- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

proof: Derivative of RHS = Integrated on LHS.

Theorem

- $\int kf(x)dx = k\int f(x)dx$
- $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

$$\frac{d}{dx}\left(\int kf(x)dx\right) = \frac{d}{dx}\left(k\int f(x)dx\right) = kf(x)$$

$$\frac{d}{dx}\left(\int f(x) \pm g(x)dx\right) = \frac{d}{dx}\left(\int f(x)dx \pm \int g(x)dx\right) = f(x) \pm g(x).$$

Example

(1) $\int 2x^5 - 3x^2 + 7x + 5dx$, $\int \frac{x^3-5}{x}dx$

(2) Find a function $F(x)$ such that $F(0) = 3$ and $F'(x) = 2x$.

Integration by Substitution

Motivating question:

$$\int (2x+1)^{10}dx = ?$$

Inconvenient to integrate by expanding the polynomial. Instead, apply integration by substitution!

Theorem

$$\int f(u(x))u'(x)dx = \int f(u)du \quad \text{or} \quad \int f(u(x))\frac{du}{dx}dx = \int f(u)du$$

$$\begin{aligned}\frac{d}{dx}\int f(u(x))u'(x)dx =& f(u(x))u'(x)\\ \frac{d}{dx}\int f(u)du =& \frac{d}{du}\int f(u)du \cdot \frac{du}{dx} \quad \text{(chain rule)}\\ =& f(u(x)) \cdot \frac{du}{dx}\end{aligned}$$

Example

$\int (2x+1)^{10}d,\quad \int e^{ax}dx,\quad \int 6x(4x^2+3)^7dx,\quad \int \frac{(\ln x)^2}{x}dx,\quad x>0.$

Question: How to make a guess of $u(x)$? Integration by Substitution:

$$\int f(u(x))u'(x)dx = \int f(u)du$$

Example: $\int \frac{(\ln x)^2}{x}dx = \int (\ln x)^2 \cdot \frac{1}{x}dx$ by letting $u = \ln x$
Realize the integrand as a product of parts and make a guess of $u(x)$.
s.t. one part can be realized as a function $f(u)$, another part is $u'(x)$.

Exercise

1) Show that $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$. (Hint: $u = ax + b$.)
2) Evaluate
a) $\int x^3 e^{x^4} dx$ (Hint: $u = x^4$)
b) $\int 6x\sqrt{x^2+3}dx$ (Hint: $u = x^2 + 3$)

Integration of exponential functions

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$$

In general: $\int a^x dx = ?$ for $a > 0$
Since $a^x = e^{\ln a^x} = e^{(\ln a)x}$

$$\int a^x dx = \int e^{(\ln a)x} dx = \frac{1}{\ln a}e^{(\ln a)x} + c = \frac{a^x}{\ln a} + c$$

Integration of logarithmic functions: $\int \ln x dx = ?$ $x > 0$
since $\frac{d}{dx}(x\ln x - x) = \ln x$!

$$\int \ln x dx = x\ln x - x + c$$

How do we know $\frac{d}{dx} x\ln x - x = \ln x$ in advance?

Integration of rational functions:

Rational Functions: $R(x) = \frac{p(x)}{q(x)}$; with polynomials $p(x), q(x)$

Simplest case: $\deg q(x) = 1$, $q(x) = ax + b$, $a \neq 0$: $\int \frac{p(x)}{ax+b} dx$ then

$$p(x) = (ax + b)u(x) + R \quad \Rightarrow \int \frac{p(x)}{ax + b} dx = \int u(x) + \frac{R}{ax + b} dx$$

Example

$\int \frac{x^2+3x+5}{x+1} dx$

Exercise

Evaluate $\int \frac{6x^2-5x+1}{3x-2} dx$

case: $\deg q(x) = 2$, i.e., $q(x) = ax^2 + bx + c$, $a \neq 0$. If $\deg q(x) \geq 2$,

$$\int \frac{p(x)}{ax^2 + bx + c} dx = \int u(x) + \frac{rx + s}{ax^2 + bx + c} dx$$

with $u(x)$ a polynomial, and only need

$$\int \frac{rx + s}{ax^2 + bx + c} dx$$

$\Delta = b^2 - 4ac$, and consider 3 subcases:

(i) $\Delta > 0$

(ii) $\Delta = 0$

(iii) $\Delta < 0$

(i) $\Delta > 0$

Then $q(x) = ax^2 + bx + c = (m_1x + n_1)(m_2x + n_2)$, and express $\frac{rx+s}{ax^2+bx+c} = \frac{A}{m_1x+n_1} + \frac{B}{m_2x+n_2}$. Then

$$\int \frac{rx+s}{ax^2+bx+c}dx = \int \frac{A}{m_1x+n_1} + \frac{B}{m_2x+n_2}dx$$

Example

$\int \frac{5x-7}{x^2-2x-3}dx$

Exercise

Evaluate $\int \frac{40}{x(200-x)}dx$

(ii) $\Delta = 0$

Then $q(x) = ax^2 + bx + c = (mx + n)^2$, and express $\frac{rx+s}{ax^2+bx+c} = \frac{A}{(mx+n)^2} + \frac{B}{mx+n}$. Then

$$\int \frac{rx + s}{ax^2 + bx + c} dx = \int \frac{A}{(mx + n)^2} + \frac{B}{mx + n} dx$$

Example

$\int \frac{2x-1}{(x-2)^2} dx$

Exercise

Evaluate $\int \frac{4x+2}{(2x-1)^2} dx$

(iii) $\Delta < 0$

$q(x) = ax^2 + bx + c$ cannot be factorized:

$$\begin{aligned}
&\int \frac{1}{x^2 + a^2} dx \quad (\text{with } x = au) \\
=&\int \frac{1}{a^2u^2 + a^2} adu \quad (dx = adu) \\
=&\frac{1}{a}\int \frac{1}{u^2 + 1} du = \frac{1}{a}\tan^{-1} u + c = \frac{1}{a}\tan^{-1}\frac{x}{a} + c
\end{aligned}$$

Example

$\int \frac{4x+7}{x^2+2x+5} dx$

General case: $\deg q(x) > 2$

Partial fraction: resolve $\frac{p(x)}{q(x)}$ into a sum of simper fractions. Then, it reduces to the above cases.

Integration of trigonometric functions:

$\int \tan x dx$ and $\int \cot x dx$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx \quad (u = cosx)$$
$$= \int -\frac{1}{u} du \quad (\frac{du}{dx} = -\sin x)$$
$$= -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c$$

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx \quad (u = \sin x) = \ln|\sin x| + c$$

$\int \sec x dx$ and $\int \csc x dx$, t-formula

let $t = \tan\frac{x}{2}$, and then express all trigonometric functions in terms of t

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}, \quad \cot x = \frac{1-t^2}{2t},$$

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},$$

$$\csc x = \frac{1+t^2}{2t}, \sec x = \frac{1+t^2}{1-t^2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(1+t^2), \quad dx = \frac{2}{1+t^2}dt$$

t-formula

let $t = \tan \frac{x}{2}$, and then express all trigonometric functions in terms of t

$$\int f(\sin x, \cos x)dx = \int \underbrace{f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}}_{\text{rational function of } t} dt$$

reduction to an integral of rational functions

t-substitution is useful for $\int \frac{1}{A\cos x + B\sin x + c}dx$

Example

$\int \csc x dx, \quad \int \sec x dx, \quad \int \frac{1}{1+\cos x}dx$

$\int \sin px \cos qx\,dx$, $\int \sin px \sin qx\,dx$, $\int \cos px \cos qx\,dx$

$$\begin{aligned} \sin px \cos qx &= \tfrac{1}{2}[\sin (p+q)x + \sin (p-q)x] \\ \cos px \cos qx &= \tfrac{1}{2}[\cos (p+q)x + \cos (p-q)x] \\ \sin px \sin qx &= -\tfrac{1}{2}[\cos (p+q)x - \cos (p-q)x]. \end{aligned}$$

Note that $\cos^2 px = \frac{1}{2}(1+\cos 2px)$ and $\sin^2 px = \frac{1}{2}(1-\cos 2px)$

Example

$\int \sin 5x \cos 3x\,dx$, $\quad \int \cos x \cos^2 3x\,dx$

Exercise

Find $\int \sin x \sin 3x \sin 6x\,dx$.

$\int \sin^m x \cos^n x dx$

Case 1: m is odd, apply $\sin x dx = -d\cos x$ and $\sin^2 x = 1 - \cos^2 x$

Case 2: n is odd, similar to Case 1, apply: $\cos x dx = d \sin x$ and $\cos^2 x = 1 - \sin^2 x$

Example

$\int \sin^4 x \cos^3 x dx$

$$= \int \sin^4 x \cos^2 x \cos x dx = \int \sin^4 x (1 - \sin^2 x) d \sin x = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$$

Case 3: m and n are even, apply:
$\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin x \cos x = \frac{1}{2}\sin 2x$

Example

$\int \sin^2 x \cos^4 x dx$

$\int \tan^m x \sec^n x dx$

Case 1: m is odd, apply: $\tan x \sec x dx = d \sec x$ and $\tan^2 x = \sec^2 x - 1$

Example

$\int \tan^3 x \sec^4 x dx$

Case 2: n is even, similar to Case 1, apply $\sec^2 x dx = d\tan x$ and $\sec^2 x = 1 + \tan^2 x$

Example

$\int \tan^4 x \sec^4 x dx$

Case 3: m is even and n is odd, use integration by parts! (later)

$\int \csc^m x \cot^n x dx$

Similarly, apply

$$\csc^2 x = -d \cot x, \quad \csc x \cot x = -d \csc x, \quad 1 + \cot^2 x = \csc^2 x$$

Exercise

Find $\int \csc^6 x \cot^4 x dx$ and $\int \csc^5 x \cot^3 x dx$

Integration of Irrational Functions

Integrand with $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{a^2-x^2}$ $(a>0)$

(1) For $\sqrt{a^2-x^2}$, we let $x = a\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(2) For $\sqrt{a^2+x^2}$, we let $x = a\tan\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(3) For $\sqrt{x^2-a^2}$, we let $x = a\sec\theta$ $\quad 0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$

Example

$\int x^3\sqrt{4-x^2}dx, \quad \int \frac{\sqrt{x^2-4}}{x^3}dx$

$\sqrt{a^2 - x^2}$ is well-defined only when $a^2 - x^2 \geq 0$, i.e., $-a < x < a$. Also $-1 \leq \sin\theta \leq 1$ for $-\frac{\pi}{2} \leq \theta\frac{\pi}{2}$. So, $-a \leq a\sin\theta \leq a$. Thus we set let $x = a\sin\theta$.

How about $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$?

Integration by parts

Let $u(x)$ and $v(x)$ be differentiable functions, then the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dv}, \quad u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

integrate both sides with respect to x:

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dv}dx$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \text{ or } \int udv = uv - \int vdu$$

Integration by parts :

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Example

$\int x^2 \ln x dx, \quad \int x^2 e^x dx$

Question: How to make a guess of $u(x)$ and $v(x)$?
Realize the integrand as a product of parts and make a guess of $u(x)$ and $v(x)$ such that one part can be realized as a function $u(x)$, another part is $v'(x)$.

Example

$\int x \sin 3x dx$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Question: Which one is $\frac{dv}{dx}$?

- 1) $u = x, \frac{dv}{dx} = \sin 3x$ $\quad \int x\sin 3x dx = \int x d(-\frac{1}{3}\cos 3x)$?
- 2) $u = \sin 3x, \frac{dv}{dx} = x$ $\quad \int (\sin 3x) x dx = \int \sin 3x d(\frac{1}{2}x^2)$?

"DETAIL" rule (not always true!)

D: dv
E: Exponential functions
T: Trigonometric functions
A: Algebraic functions (roots of polynomials e.g. $x^3, \sqrt{x+1}$)
I: Inverse trigonometric functions
L: Logarithmic functions

"DETAIL" order of choosing $\frac{dv}{dx}$

Integration of log functions

$\int \ln x dx$

Integration of inverse trigonometric functions

$\int \tan^{-1} x dx = x \tan^{-1} x - \int x d \tan^{-1} x$

Exercise

By integration by parts, show $\int \sin^{-1} x dx = \sqrt{1 - x^2} + x \sin^{-1} x + c$ and $\int \cos^{-1} x dx = \sqrt{1 - x^2} + x \cos^{-1} x + c$

transformed into the original integral

Example

- $\int e^x \cos x dx = \int e^x d \sin x$,
- $\int \sin(\ln x) dx = x \sin(\ln x) - \int x d \sin(\ln x)$,
- $\int \sec^3 x dx$