MATH1010G University Mathematics
Week 8: Taylor series and L'Hospital's rule

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Indeterminate form $\frac{0}{0}$ and L'hospital's rule

Example (NOT RIGOROUS)

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

It suggests

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

How do we know $\lim_{x \to 0} -\frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = 0$? There are infinitely many terms!

In general,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots$$

Suppose, $f(a) = g(a) = 0$ and $f'(a), g'(a) \neq 0$, then

$$\begin{aligned} \lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots}{g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots} \\ &= \lim_{x\to a}\frac{f'(a) + \text{terms involves } (x-a)}{g'(a) + \text{terms involves } (x-a)} = \frac{f'(a)}{g'(a)}. \end{aligned}$$

Consider $\lim_{x\to a}\frac{f(x)}{g(x)}$ and suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

- Case 1: If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
- Case 2: If $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x) \neq 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ does NOT exist. (e.g., $\lim_{x\to 1}\frac{x}{x-1}$)
- Case 3: If $\lim_{x\to a} g(x) = 0 = \lim_{x\to a} f(x) = 0$, then we do NOT know whether $\lim_{x\to a}\frac{f(x)}{g(x)}$ exist ?

Example

$$\lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0} x = 0,$$

$$\lim_{x\to 0}\frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1,$$

$$\lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0}\frac{1}{x} \text{ does not exist.}$$

called indeterminate form $\frac{0}{0}$.

L'hospital's rule

Suppose that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, I is an open interval containing a, f and g are differentiable on $I \setminus \{a\}$, and $g'(x) \neq 0$ on $I \setminus \{a\}$. If $\lim \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example

$$\lim_{x\to 0} \frac{\sin x}{x} \qquad \left(\frac{0}{0}\right)$$

Example

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \qquad \left(\frac{0}{0}\right)$$

Example (L'hospital's rule fails)

$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} \quad \left(\frac{0}{0}\right)$

$$\lim_{x\to 0} \frac{2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})}{\cos x}$$

and this limit does no exist. However,

$$\lim_{x\to 0} \frac{x^2\sin(\frac{1}{x})}{\sin x} = \lim_{x\to 0} \frac{x\sin(\frac{1}{x})}{(\frac{\sin x}{x})} = \frac{0}{1} = 0.$$

Indeterminate form $\frac{0}{0}, \infty \cdot 0, \infty - \infty$

- L'hospital's rule applies to $\frac{\infty}{\infty}$
- L'hospital's rule applies to the left-hand limit or right-hand limit.

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}, \quad \lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{x \to a^-} \frac{f'(x)}{g'(x)}$$

- L'hospital's rule applies to the limits at infinities.

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}, \quad \lim_{x \to -\infty} \frac{f(x)}{g(x)} = \lim_{x \to -\infty} \frac{f'(x)}{g'(x)}$$

Example

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\sec x}{1+\tan x}\qquad\left(\frac{\infty}{\infty}\right)$$

Example

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}\qquad\left(\frac{\infty}{\infty}\right)$$

For indeterminate form $\infty \cdot 0$, converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example

$$\lim_{x\to+\infty} x\sin\frac{1}{x} \quad (\infty\cdot 0)$$

Example

$$\lim_{x \to 0^+} \sqrt{x} \ln x \quad (\infty, 0)$$

For indeterminate form $\infty - \infty$, converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example

$\lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right) \quad (\infty - \infty)$

$(\infty - \infty)$ convert to $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0^+} \frac{x - \sin x}{x \sin x} = 0$$

Indeterminate form $1^\infty, 0^0, \infty^0$: taking ln, converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example

Find $\lim\limits_{x\to 1^+} x^{\frac{1}{1-x}}$ $\qquad (1^\infty)$

Example

Find $\lim_{x \to +\infty} x^{\frac{1}{x}}$ $\qquad (\infty^0)$

Exercise

Show that $\lim_{x \to +\infty} (1 + \frac{1}{x})^x = e$ by L'hospital's rule.