MATH1010G University Mathematics
Week 6: Differentiation (continued)

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February 2024

review

- definition of derivatives
- differentiability $\Rightarrow$ continuity, high-order derivative
- product rule / quotient rule
- derivative of trigonometric / exp functions
- chain rule

Find the derivatives of the following functions

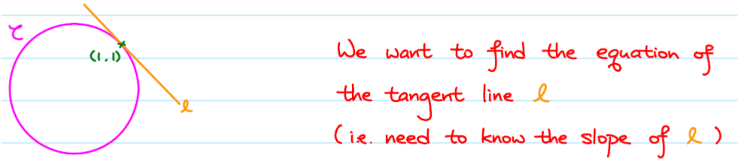
- $f(x) = \frac{\sin(x)}{x}$
- $f(x) = \frac{e^{\sin x}}{\cos x}$
- $f(x) = \sqrt{1 - \sin(\cos x^2)}$
- $f(x) = \cos \frac{1}{x}$

implicit differentiation

Example

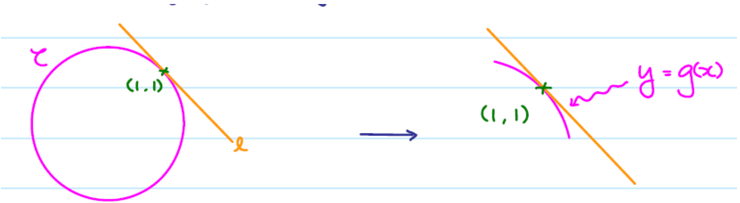
$$x^2 + y^2 = 2 \to \mathcal{C}.$$

Locus of $\mathcal{C}$ is a circle centered at $(0, 0)$ with radius $\sqrt{2}$.
check: $(1, 1)$ is a point lying on the circle.



$x^2 + y^2 = 2$ is NOT a function!

Question: How to find $\frac{dy}{dx}$? (actually, is it well defined?)



The small segment of $\mathcal{C}$ containing $(1, 1)$ can be regarded as the graph of some function $y = g(x)$. (In fact, $g(x) = \sqrt{2 - x^2}$.)

How to find it? Do it as usual!

$$x^2 + y^2 = 2$$

differentiate both sides with respect to x, and apply chain rule.

$$2x + \frac{d}{dx}y^2 = 0, \quad 2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Therefore, $\frac{dy}{dx} = -1$ when $(x, y) = (1, 1)$, i.e.,

$$\frac{dy}{dx}|_{(x,y)=(1,1)} = -1$$

$\frac{dy}{dx}$ is defined at a point of a curve only if a small arc containing the point can be regarded as the graph of some function $y = g(x)$.
Hence $\frac{dy}{dx}$ is NOT defined when $(x, y) = (\pm\sqrt{2}, 0)$.

implicit differentiation

Apply differentiation to $F(x, y)$=0

$x^2 + y^2 = 2 \rightarrow F(x, y) = x^2 + y^2 - 2 = 0$

Example

Let $\mathcal{C}$ be the curve defined by the equation $x^3 + 2y^3 + 2xy = 5$.

- Show that $P = (1, 1)$ is a point lying on $\mathcal{C}$.
- Find the equation of the tangent line of $\mathcal{C}$ at P.

Differentiation of logarithmic function.

Example

Let $y = \ln x$, $x > 0$.

Exercise

show $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$.

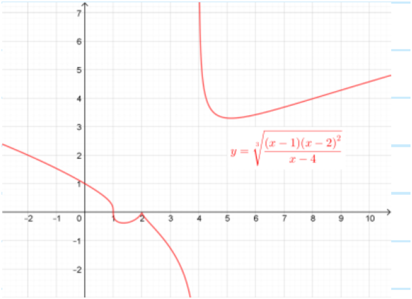
Note: $\log_a x = \frac{\ln x}{\ln a}$

Example

Let $y = \ln|x|$, $x \neq 0$. Find $\frac{dy}{dx}$.

Example

If $y = \sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}$, then find $\frac{dy}{dx}$.



Difficult to differentiate by using the chain rule and quotient rule!

Example

Let $y = \frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$. Find $\frac{dy}{dx}$.

differentiation of inverse trigonometric functions.

Example

Let $y=\sin^{-1} x$, $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$. Show $y' = \frac{1}{\sqrt{1-x^2}}$

Example

Let $y = \cos^{-1} x$, $\cos^{-1} : [-1,1] \to [0,\pi]$. Show $y' = -\frac{1}{\sqrt{1-x^2}}$

Exercise

Let $y = \tan^{-1} x$, $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Find $\frac{dy}{dx}$.

(hint: $\frac{d}{dx} \tan x = \frac{1}{\sec^2 x}$)

Example

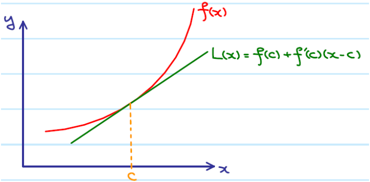
Let $y = x^x$ for $x > 0$. Find $\frac{dy}{dx}$.

The power is NOT a constant, we cannot use the formula $\frac{d}{dx} x^n = nx^{n-1}$.

Let $f(x)$ be a function which is differentiable at $x = c$. The graph of $f(x)$ passes through the point $(c, f(c))$ and the slope of the tangent line of $f(x)$ at x=c is $f'(c)$. So the tangent line is given by

$$y - f(c) = f'(c)(x - c), \quad y = f(c) + f'(c)(x - c)$$

Idea: The graph of $L(x)$ is close to the graph of $f(x)$ around $x = c$, so $f(x)$ can be approximated by $L(x)$ around x=c.



$L(x) = f(c) + f'(c)(x - c)$ is said to be the linearization of $f(x)$ at x=c.

Example

Find the linearization of $f(x) = \sqrt{x}$ at $x = 100$ (and hence approximate $\sqrt{101}$).

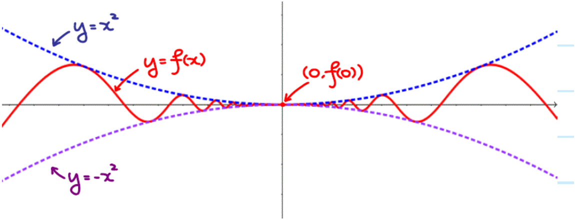
More on differentiability

Example

Let

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Does $f'(0)$ exist?



Exercise

Show $\lim_{x\to 0} f'(x)$ does NOT exist.

Hence f is diff. ("good" in some sense) but $f'(x)$ can be "bad".

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant function such that

(i) f is differentiable at some $x_0 \in \mathbb{R}$

(ii) $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Show that:

- $f(x) \neq 0$ for all $x \in \mathbb{R}$ and $f(0) = 1$.
- f is differentiable at every $x \in \mathbb{R}$ and $f'(x) = \frac{f'(x_0)}{f(x_0)} f(x)$.

Exercise

Let f be a differentiable function such that

$$f(x+y) = f(x) + f(y) + 3xy(x+y) \quad \forall x, y \in \mathbb{R}.$$

(a) Show that $f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}$.

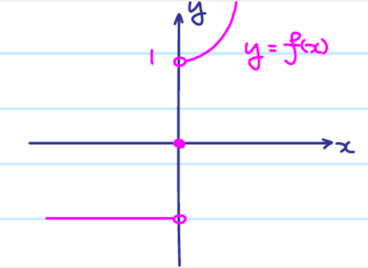
(b) Show that $f'(x) = f'(0) + 3x^2$.

(In fact, $f(x) = c + f'(0)x + x^3$ if you know integration.)

Exercise 5.10.2

Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$



(a) Write down $f'(x)$ explicitly.
(b) Show that f is not differentiable at $x = 0$.
(c) Show that $\lim_{x\to 0^-} f'(x) = 0$ and $\lim_{x\to 0^+} f'(x) = 0$. so $\lim_{x\to 0} f'(x) = 0$.

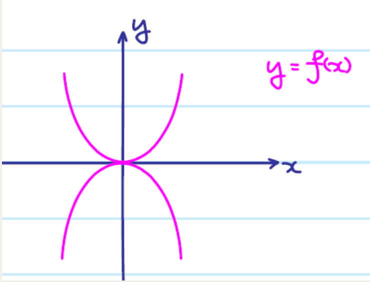
Therefore, $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$ is insufficient to show f is differentiable at $x = 0$.

Exercise

Let

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ -x^2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that $f(x)$ is differentiable only at $x = 0$ and $f'(0) = 0$.



Therefore, $f'(0)$ exists while $\lim_{x \to 0^-} f'(x)$, $\lim_{x \to 0^+} f'(x)$ are not.

Summary

f is differentiable at $x = x_0$, does NOT imply

a) f' is differentiable at $x = x_0$.

b) f' is continuous at $x = x_0$.

c) $\lim_{x \to x_0} f'(x)$ exists.

Summary

f is differentiable at $x = x_0$, implies:

a) f is continuous at $x = x_0$.

b) $\lim\limits_{x \to x_0} f(x)$ exists.

But the converse may be false!

Summary

f is differentiable at $x = x_0$ does NOT imply $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^+} f'(x)$.

(Also, the converse is not true.)

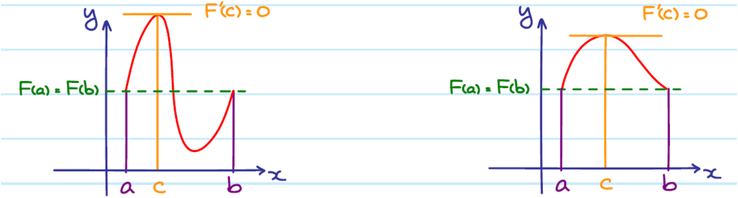
Rolle's Theorem and Mean Value Theorem

Theorem

Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$ such that (i)$f'(c)$ exists. (ii) f attains maximum (or minimum) at $x = c$. Then, $f'(c) = 0$.

Rolle's theorem

Let $F : [a, b] \to \mathbb{R}$ be a function such that (i) F is continuous on $[a, b]$. (ii) F is differentiable on (a, b). (iii) $F(a) = F(b)$. Then there exists $c \in (a, b)$ such that $F'(c) = 0$.



Idea of proof: By the Maximum-Minimum Theorem, there exists $x_m, x_M \in [a, b]$ such that $F(x_m) \leq F(x) \leq F(x_M)$ for all $x \in [a, b]$.

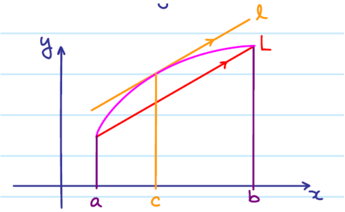
- Either x_m or x_M lies on (a, b) then $F'(x_m) = 0$ or $F'(x_M) = 0$
- Both x_m and x_M lies on boundary points $[a, b]$. By assumption, $F(a) = F(b)$ which forces that $F(x)$ is constant on $[a, b]$ so $f'(x) = 0$ for all $x \in (a, b)$.

Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function such that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

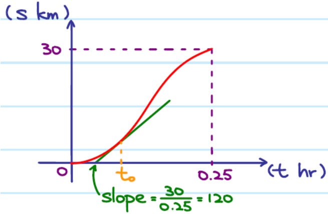
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). MVT implies there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. In particular, if $f(a) = f(b)$, then $f'(c) = 0$ (i.e., Rolle's theorem). Hence, Rolle's theorem is a special case of MVT.



Question

A vehicle is speeding on a highway if its speed $\geq$ 120 km/hr (at some moment). If the length of the highway is 30 km and if a driver only spent 15 minutes on the highway. Should he be arrested?



By the MVT, there exists $t_0 \in (0, 0.25)$ such that slope of the tangent at $t = t_0$ $= \frac{30}{0.25} = 120$

i.e. instantaneous speed at $t = t_0$ $= 120$ km/hr

Example

Note $\ln 1 = 0$, but how about $\ln 1.1$?

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable and $f'(x) = 0$ for all $x \in \mathbb{R}$, then $f(x)$ is a constant function.

Example

Let $f(x)$=$\cos^2 x + \sin^2 x$.

Theorem

If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in \mathbb{R}$, then $f(x) = g(x) + c$, where c is a constant.

Let $h(x) = f(x) - g(x)$. Then

$$h'(x) = f'(x) - g'(x) = 0.$$

Therefore, $h'(x) = c$, where c is a constant, i.e., $f(x) = g(x) + c$.

how differentiation helps to find maximum/minimum ?

Definition

Let I be an interval and let $f : I \to \mathbb{R}$ be a function such that

$$f(x_1) \leq f(x_2) \quad (f(x_1) \geq f(x_2)), \quad \forall x_1 < x_2.$$

Then $f(x)$ is called an increasing (a decreasing) function. If we have a strictly inequality, it is called a strictly increasing (decreasing) function.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b) and $f'(x) \geq 0$ ($f'(x) \leq 0$) for all $x \in (a, b)$.

Then f is increasing (decreasing) on $[a, b]$. If we have strict inequality, then $f(x)$ is strictly increasing(decreasing) on $[a, b]$.

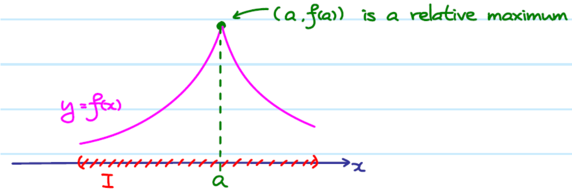
Theorem (first derivative check)

Let I be an open interval, $a \in I$, and $f : I \to \mathbb{R}$ be a function such that

- f is continuous
- $f'(x) \geq 0 \quad (f'(x) \leq 0)$ for all $x \in I$ with $x < a$.
- $f'(x) \leq 0 \quad (f'(x) \geq 0)$ for all $x \in I$ with $x > a$.

Then $(a, f(a))$ is a relative maximum (minimum).

We do NOT require the differentiability of of f at $x = a$, but only the continuity of f at $x = a$.



Since f is continuous on $[x, a]$ and differentiable on (x, a), applying MVT to f on $[x, a]$, $\exists c \in (x, a)$ such that

$$f(a) - f(x) = f'(c)(a - x) \geq 0$$

Similarly, $f(x) \leq f(a)$ for all $x \in I$ with $x > a$.

Definition

If $f'(a) = 0$, then $(a, f(a))$ is said to be a stationary point.

Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$ such that

- $f'(c)$ exists.
- f attains maximum (or minimum) at $x = c$.

Then, we have $f'(c) = 0$.

Implication: If $f(x)$ is differentiable everywhere, then all maximum / minimum points are stationary. However, a stationary point is NOT necessary to be a maximum and minimum point!

Example

If $f(x) = x^3$ then $f'(x) = 3x^2$.

Example

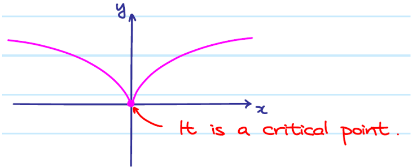
Let $f(x) = \sqrt{|x|}$

Note that: f is continuous at $x = 0$, by the first derivative check, $f(x)$ attains minimum at $x = 0$.

Definition

If $f'(a) = 0$ or $f'(a)$ does not exist, then $(a, f(a))$ is a critical point.

a stationary points is a critical point.



Example

Prove that $e^x \geq 1 + x$ (i.e., $e^x - x - 1 \geq 0$) for all $x \in \mathbb{R}$.

Exercise

Prove that $1 - \frac{1}{x} \leq \ln x \leq x - 1, \quad \forall x > 0$

Exercise

Fix $\alpha \in (0, 1)$. Prove that

$$1 + \alpha x - \frac{\alpha(1-\alpha)}{2} x^2 < (1+x)^\alpha < 1 + \alpha x, \quad \forall x > 0$$

Exercise

(a) Show $f(x) = x^{\frac{1}{3}} - \frac{1}{3}x - \frac{2}{3} \leq 0$ for all $x > 0$.

(b) Show $u^{\frac{1}{3}} v^{\frac{2}{3}} \leq \frac{1}{3}u + \frac{2}{3}v$ for all $u, v > 0$ (Hint: set $x = \frac{u}{v}$).

Example

Find the max / min of $f(x) = x^3 - 3x^2 - 9x + 5$

Second and higher derivatives

Definition

Given $y = f(x)$, the second derivative of $f(x)$ is the function

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The second derivative of $y = f(x)$ is also denoted as $f''(x)$ or y''. Let n be a nonnegative integer. Then the nth derivative of $y = f(x)$ is defined inductively by

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right), \quad n \geq 1,$$
$$\frac{d^0y}{dx^0} = y$$

The nth derivative is also denoted as $f^{(n)}$ or $y^{(n)}$.

Example

Find $\frac{d^2y}{dx^2}$ for

- $y = \ln(\sec x + \tan x)$
- $x^2 - y^2 = 1$

Theorem (Leibniz's rule)

Let u and v be differentiable functions of x. Then with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$,

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

Example

- $(uv)^{(0)} = uv$
- $(uv)^{(1)} = u'v + uv'$
- $(uv)'' = u''v + 2u'v' + uv''$
- $(uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$
- ...

Example

Let $y = x^2 e^{3x}$. Find $y^{(n)}$.

summary

- implicit differentiation
- linearization
- more on differentiability & continuity
- Rolle's theorem and mean value theorem
- monotonicity and first derivative check (relative max / min)
- Leibniz's rule