MATH1010G University Mathematics
Week 5: Differentiation

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February 2024

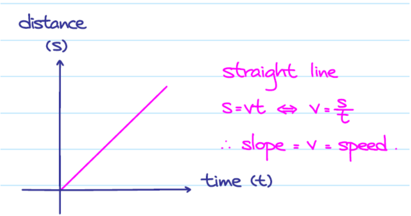
review

- limits of functions at infinity
- special limits, algebraic properties, sandwich theorem
- continuity
- relative / absolute extremum (maximum / minimum)

Derivative

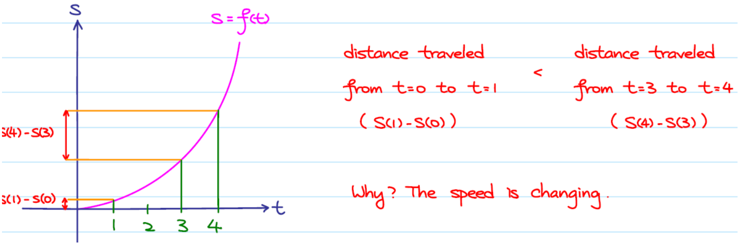
Idea of derivative:

$$\text{(average) speed} = \frac{\text{distance}}{\text{time}}.$$



Using displacement and velocity if you know.

How about this case?



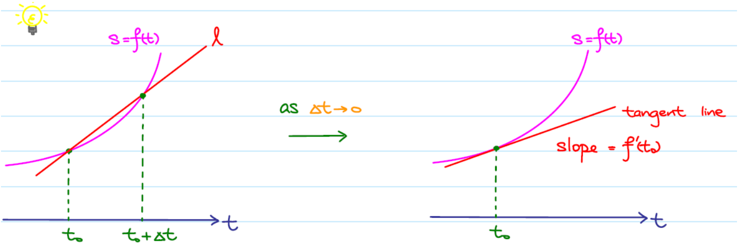
Speed is different at different moments.

Hold on! What is the meaning of speed at a particular moment (instantaneous speed)?

idea: let Δt become smaller and smaller, and instantaneous speed at $t = t_0$ is defined to be

$$\lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$$

(provided that it exists, if so, it is denoted by $f'(t_0)$).



When $\Delta t \to 0$, ℓ tends to the tangent line at $t = t_0$, so the slope of the tangent line at $t = t_0 = f'(t_0)$.

Example

If $f(t) = t^2$, find $f'(2)$ (instantaneous speed at $t = 2$).

Let $y = f(x)$, fix x_0. Then $f'(x_0)$ is the rate of change of y with respect to x at $x = x_0$.

Definition

$f(x)$ is differentiable at $x = x_0$ if

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists. It is called the derivative of f at $x = x_0$, and denoted by $f'(x_0)$.

Note: by definition, if $f(x_0)$ is NOT well-defined, then $f'(x_0)$ is NOT well-defined.

Let $\Delta x = x - x_0$, i.e. $x = x_0 + \Delta x$. When Δx tends to 0, x tends to x_0.

$\Rightarrow f(x)$ differentiable at $x = x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists}$$

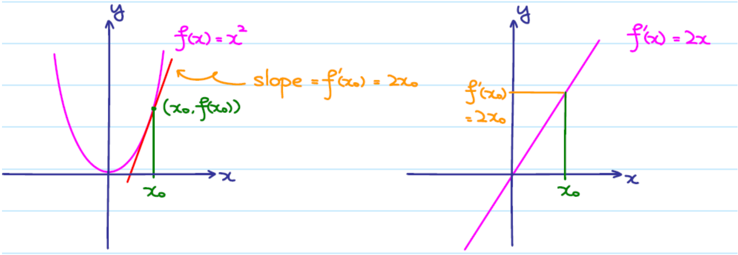
A function, roughly speaking, given an input x, return a value. Now, we construct a new function,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{(if exists)}$$

(Given an input x, return the slope of the tangent line at $(x, f(x))$)

Example 5.1.2

If $f(x) = x^2$, find $f'(x)$.



Notations

$$
\begin{aligned}
y = f(x) &= x^2 \\
\frac{df}{dx} = \frac{dy}{dx} &= f'(x) = 2x \\
\left.\frac{df}{dx}\right|_{x=3} = \left.\frac{dy}{dx}\right|_{x=3} &= f'(3) = 2(3) = 6.
\end{aligned}
$$

Definition

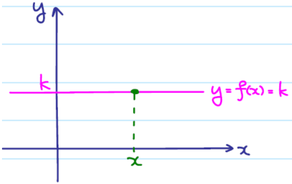
If $f : D \to \mathbb{R}$ is differentiable at every point in D, then f is said to be a differentiable function.

Exercise

Find $f'(x)$ for $f(x) = x$ and $f(x) = x^3$.

Theorem

If $f(x) = k$, where k is a constant, then $f'(x) = 0$.



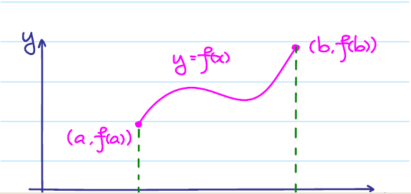
Note : tangent line at $(x, f(x))$ is horizontal

$\therefore f'(x) = 0$

Definition

Let $f : [a, b] \to \mathbb{R}$.

- f is said to be differentiable at $x = a$ if $\lim_{\Delta x \to 0^+} \frac{f(a+\Delta x) - f(a)}{\Delta x}$ or $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$ exists.
- f is said to be differentiable at $x = b$ if $\lim_{\Delta x \to 0^-} \frac{f(b+\Delta x) - f(b)}{\Delta x}$ or $\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$ exists.

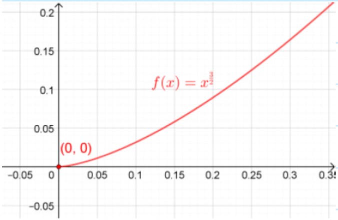


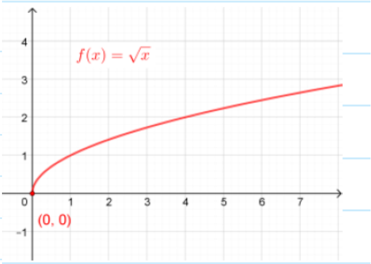
Example

Let $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^{\frac{3}{2}}$.

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^{\frac{3}{2}}}{x} = \lim_{x \to 0^+} x^{\frac{1}{2}} = 0.$$

Hence f is differentiable at $x = 0$.





Example

Let $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$.

f is not differentiable at $x = 0$

Theorem

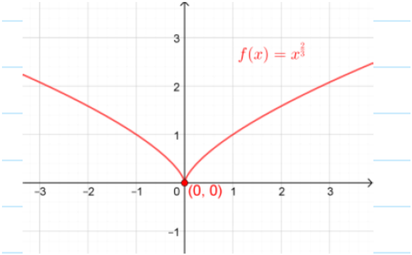
If $f(x) = x^r$, with $r \in \mathbb{R}$, then $f'(x) = rx^{r-1}$, whenever it is defined.

only prove the case $f(x) = x^n$, where n is the positive integer

Exercise

Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{\frac{2}{3}}$.

- Show that f is not differentiable at $x = 0$.
- Show that $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ for all $x \neq 0$.



Differentiability and continuity

Theorem

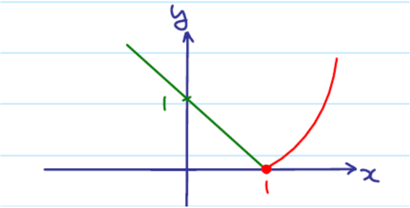
If $f(x)$ is differentiable at $x = x_0$, then $f(x)$ is continuous at $x = x_0$.

Example

Let

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x \leq 1, \end{cases}$$

is f differentiable at $x = 1$?



Notation:

$Lf'(x_0) = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$ (left derivative of f at $x = x_0$)

$Rf'(x_0) = \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$ (right derivative of f at $x = x_0$)

Then, f is differentiable at $x = x_0$ if and only if $Lf'(x_0) = Rf'(x_0)$.

Exercise

- Show that $f(x)$ is continuous at $x = 1$, i.e., $\lim_{x \to 1} f(x) = f(1)$. (Therefore, the converse statement of the theorem is NOT true.)
- Write down $f'(x)$ for $x \neq 1$.

Differentiation rules

Theorem

If $f(x)$ and $g(x)$ are differentiable functions, then

- $(f+g)'(x) = f'(x) + g'(x)$
- $(f-g)'(x) = f'(x) - g'(x)$
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$ **[product rule]**
- $(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$. **[quotient rule]**

Corollary

If k is a constant and $f(x)$ is diff., then $(k \cdot f)'(x) = kf'(x)$.

Using the product rule:

Example

Find $\frac{d}{dx}(3x^2 + 7x - 2)$.

Example

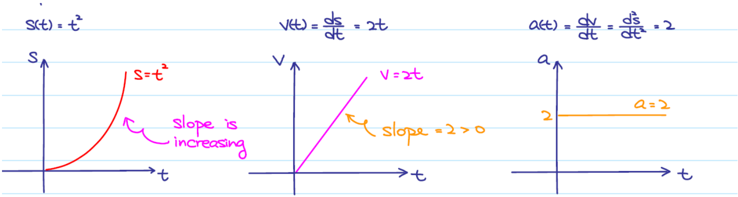
Find $\frac{d}{dx}[(3x^2 - 5x + 1)(2x + 7)]$.

Example

Find $\frac{d}{dx}\frac{2x}{x^2+1}$, $\frac{d}{dx}(\frac{1}{\sqrt{x}} + \sqrt{x})$.

Higher derivatives

- $s(t)$: distance functions (depends on time t)
- (instantaneous) speed = rate of change of distance traveled with respect to t.
  $v(t) = \frac{ds}{dt}$ (still a function of t.)
- Question: What is $\frac{dv}{dt}$?
  acceleration = rate of change of speed with respect to t
  $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.



In general, let $y = f(x)$.

$$\begin{aligned}
&(1st \quad \text{derivative}) \quad \frac{dy}{dx} = \frac{df}{dx} = f'(x) \\
&(2nd \quad \text{derivative}) \quad \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x) \\
&(nth \quad \text{derivative}) \quad \frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = f^{(n)}(x).
\end{aligned}$$

$$\frac{d}{dx}\cos x = -\sin x.$$

Exercise

Prove $\frac{d}{dx}\sin x = \cos x$, and $\frac{d}{dx}\tan x = \sec^2 x$.

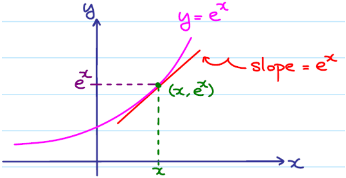
Exercise

Prove $\frac{d}{dx}\sec x = \sec x \tan x$, $\frac{d}{dx}\csc x = -\csc x \cot x$, $\frac{d}{dx}\cot x = -\csc^2 x$

Derivative of e^x

$$\text{Cheating}: e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\begin{aligned}\frac{d}{dx}e^x &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\\ &= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\\ &= e^x.\end{aligned}$$



The derivative of e^x is itself.

Example

Find $\frac{d}{dx}\left[e^x(3x^2+7x-2)\right]$.

Chain rule

Question: How do we differentiate composite functions ?

Theorem

If $f : B \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are differentiable such that $g(A) \subset B$, then the composition $(f \circ g) : A \to \mathbb{R}$ is differentiable and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Interpretation: Let $u = g(x), y = f(u) = f(g(x))$, then the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Think : $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$

Example

Find the derivative of $\sqrt{x^2 + 3x}$.

Example

Find the derivative of $(3x^2 - 2x)^{10}$.

Exercise

Show that $\frac{d}{dx}e^{ax} = ae^{ax}$, and $\frac{d}{dx}a^x = (\ln a)a^x$, for $a > 0$.

(Hint: $a^x = e^{(\ln a^x)} = e^{(\ln a)x}$)

Exercise

Find the derivative of $(\frac{x}{x+1})^2$.

- by using the chain rule
- by writing $(\frac{x}{x+1})^2 = \frac{x^2}{(x+1)^2}$ and using the quotient rule.

Example

Find the derivative of $e^{\sqrt{x^2+1}}$.

Example

Revisit of quotient rule

summary

- definition of derivatives
- differentiability $\Rightarrow$ continuity, high-order derivative
- product rule / quotient rule
- derivative of trigonometric / exp functions
- chain rule